Introduction to CFD
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Lecture – 26
Numerical Solution of Unsteady Heat Conduction (Parabolic PDE) (continued)

In this lecture, we will start our discussion with the Crank-Nicolson scheme for discretizing unsteady heat conduction equation.

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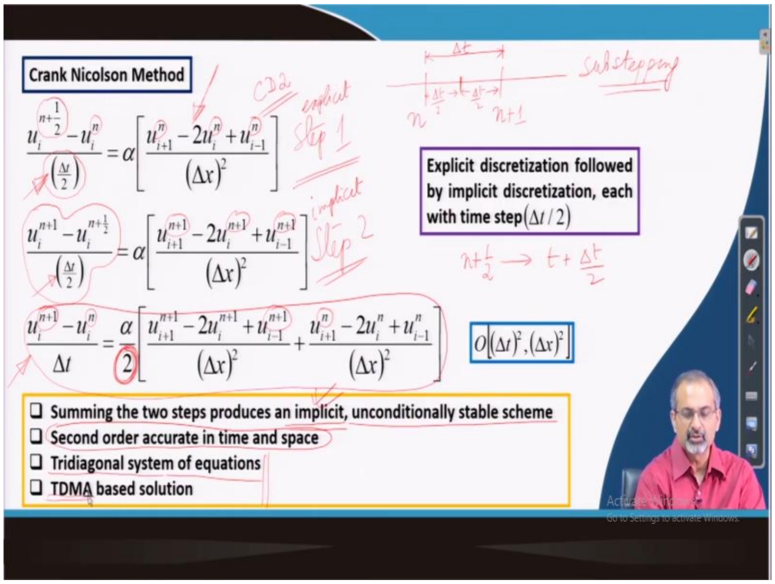

Let us look at, how the Crank-Nicolson scheme or method works. We have essentially a 2 step discretization process which forms the Crank-Nicolson scheme as you can see above. So, this is like step 1 and this is step 2, if we would like to call them that way. So, in the step 1, what happens is, we use a delta t by 2 time step. So, if we look at the time axis, what do we have if we are at the nth time interval.

And if we are moving to the n + 1 at time interval, one way of going to the next time interval is to use the time step delta t directly. Another way of doing it could be by doing what is called as sub stepping. That means, we are dividing the time step delta t into some sub steps. So, in this case, we are using 2 sub steps each of duration delta t by 2. So, in the first step, what we are doing is we are progressing the solution from the nth time step to the n plus half time step.

And because that involves a delta t by 2 time step, we divide the difference in u by delta t by 2 on the left hand side to approximate the time derivative. When it comes to approximating the space derivative, we use the CD2 scheme as usual, for discretizing, the space derivative, the second order derivative, and we use the nth time step for their values. So, this essentially is an explicit step. This step is followed by an implicit step.

So, if we look at the step 2, it is essentially an implicit step. Why? because now, on the left hand side, of course, you are stepping it from n plus half, which essentially stands for n plus half would stand for t + delta t by 2. So, we are moving from n plus half to n + 1. Again, it involves delta t by 2 times stepping, so, therefore, it is divided by delta t by 2. And on the right hand side, again CD2.

But, because it is done for the next time step n + 1, this is an implicit calculation. Now, if you were to sum of these 2 steps, what it would produce is the form that you have at the bottom. And this is essentially the standard form of the Crank-Nicolson method. So, this is nothing but a summation of those 2 above steps. And what you have as a consequence is that you have a n and n + 1 here on the left hand side, the n plus half has vanished off.

Because it has got cancelled when you added up those 2 equations, and what do you have on the right hand side are central differencing involving the 2 time steps in an n + 1. So, because you are kind of using two half steps of time, you are seeing a division by 2 coming up over here. Now, this essentially produces an implicit which is an unconditionally stable scheme. More importantly, it gives you second order accuracy in time and space.

So, if you remember the Lasonaz scheme had given us first order accurate in time and second order accurate in space, but here we have second order accuracy overall in time and space. We already discussed about the big advantage of having unconditional stability due to the implicit nature of the scheme and if you were to write down this equation by segregating the unknowns and knowns.

You will be able to see the tridiagonals structure which can be used, which can be solved using TDMA.

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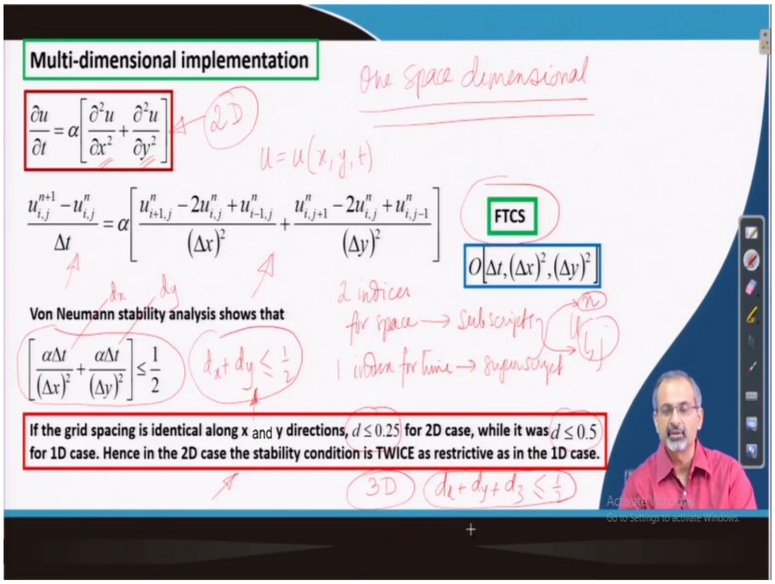

We have discussed about only one space dimension mostly when we talked about parabolic partial differential equations now, let us have a look at multi-dimensional applications. So, here we are talking about two dimensional space involving x and y. So, u now is a function of x, y and t. And here what we have done is to discretize the two dimensional form of the transient heat conduction equation.

We have used the FTCS way of discretizing the time and space derivatives. So, this is forward in time for strata accurate and this is central in space second order accurate, but we are involving 2 directions and therefore, 2 indices for space, one index for time. So, the 2 indices for space occur as subscripts and this as a superscript. So, if you have a term u i j, so, these are the 2 subscripts here and the n is the superscript.

Now, if you are concerned about the stability, because this is an explicit scheme that we are talking about, then of course, you have to go back and do the Von Neumann stability analysis. And then Von Neumann stability analysis says that the stability condition would be something like this. So, these are essentially the diffusion numbers along the 2 directions x and y. Say, if you call them dx and dy, of course, in a suffix sense.

Then a summation of those two would be less than or equal to half that means individually both of them would have to be smaller than what it is for a corresponding one dimensional case. And that is a big worry. So, if you were to use the same grid spacing along the 2 directions, then this condition produces an outcome that the diffusion number has to be less than equal to point 25 for a two dimensional case whereas it was less than equal to point 5 for a one dimensional case.

No, what does that mean? That means, that for the same grid spacing, you now have 2 time step the solution using half the time steps that you could have used for a one dimensional case, which means your simulations become that much slower. First of all, the simulations would slow down because you have 2 space dimensions now to cover.

Additionally, you can now time step more slowly, half as slowly as the one dimensional case. Therefore, the simulations will get increasingly slower. You can imagine how the situation would further degrade. If you were to think about a three dimensional situation, because you can then anticipate, you would have a situation like dx + dy + dz, all suffixes x y z, less than equal to half, which means if all the grid spacing’s are equal.

Then you would have a still smaller d allowed for a three dimensional space. How would one go about doing it for a situation like this? So, let us try to get a feel.

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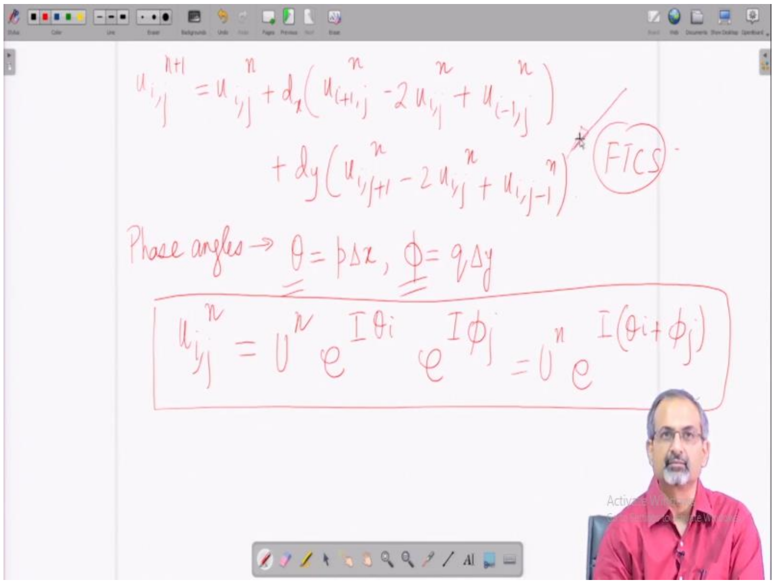

So, we are just trying to write down the equation once again using the diffusion numbers. So, this is how the discretized form would look like using the FICS scheme. Now, in a multi-dimensional situation, what we need to do is we have to introduce phase angles along the respective directions. So, what would we do for the phase angles? So, along x, if we use the theta nomenclature like we were doing earlier.

We can call it as say p times delta x, where p is the wave number along x. And let us introduce a phase angle nomenclature phi along the y direction that is called the wave number q along the y direction. So, we have to introduce separate wave numbers along the separate directions in a multi-dimensional situation in order to define the corresponding phase angles. Once we do that then the approach remains very similar to what we did earlier.

Let us try to write down the expression now for a term like u i j n. So, the amplitude term and then you now have 2 exponential terms to accommodate the 2 directions and the corresponding phase angles coming up there. So, what you can do is you can simplify the exponential term in this manner, in this form. So, once you do that then this kind of methodology can be used for all the terms that you have in the FTCS discretization.

So, let us do it. So, this takes care of the x direction, and then we continue along the y direction with the dy coming outside the bracket.

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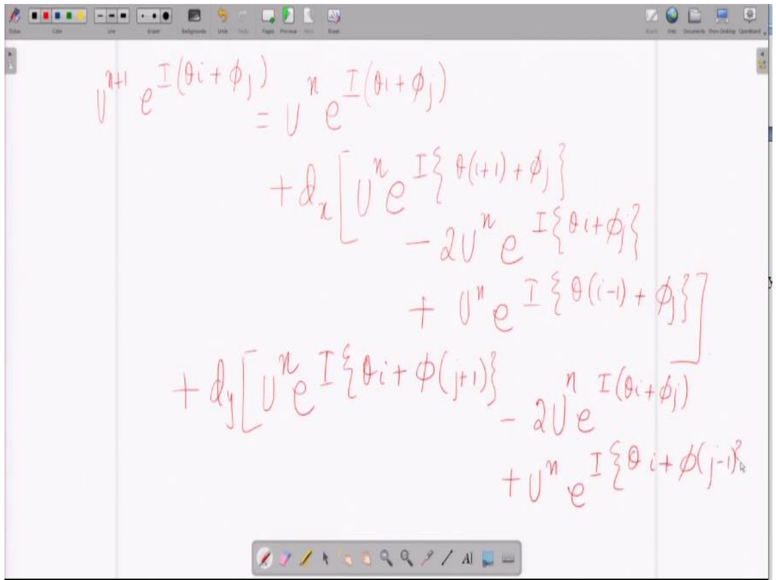

And then the; rest of the terms coming from the spatial derivative along y coming inside the bracket. So, here the terms with involving phi would change and theta part would remain the same. So, this is how it will look like. Now, we need to factor out u n from the bracketed terms on the right hand side as well as e to the power of i and in brackets theta i + phi j.

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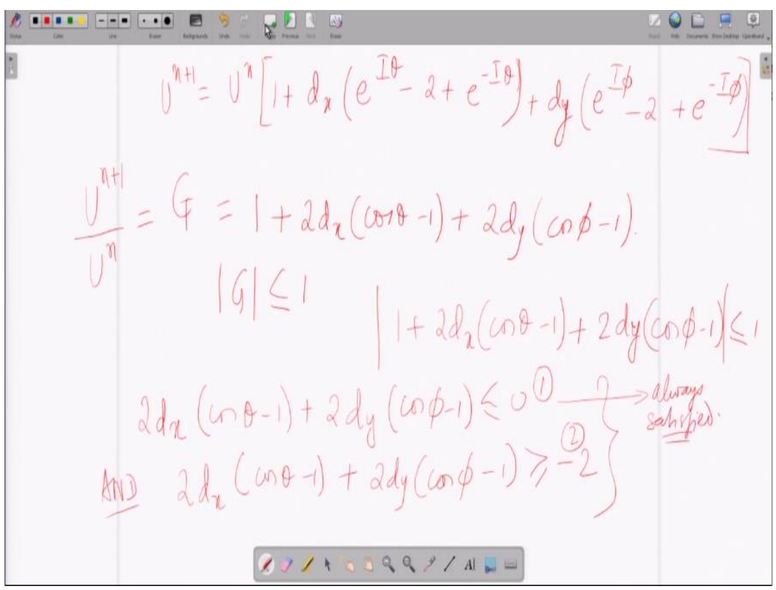

So these would be the common terms which can be factored out. Once you do that it would look simpler. This is how it will look like. And then once you divide u n + 1 by u n, so, we divide both the sides of the equation by u n in order to get the amplification factor. So, the amplification factor would come out to be like this. And we know that for a stable solution, this needs to be less than equal to 1.

So, that essentially means that mode of this expression should be less than equal to 1. And what that effectively means is that this is less than equal to 0. And so, these 2 conditions have to be fulfilled. Now, we can show very easily for the entire range of values of cos theta and cos phi satisfying the condition one is not an issue because it is always satisfied. But, this condition 2 would not always be satisfied and that is what brings in the condition that.

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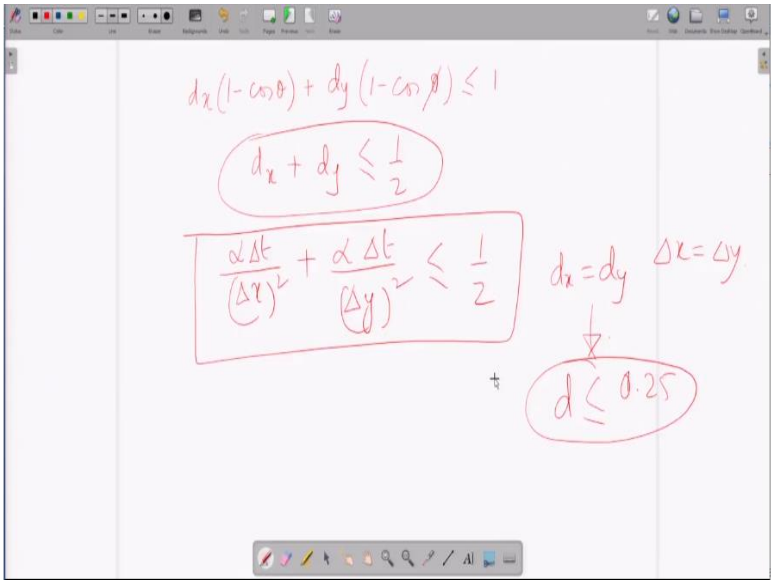

And when we use the extreme values of cos theta and cos phi, we can show that this condition will come out and that is what we had written earlier that alpha delta t by delta x square + alpha delta t by delta y square is less than equal to half and when dx is equal to dy which will happen when delta x is equal to delta y. Then the outcome is that the d is less than equal to point 25 which is twice as restrictive as compared to what it was in one dimensional case. So, this is what we intended to show and we have come up with the outcome.

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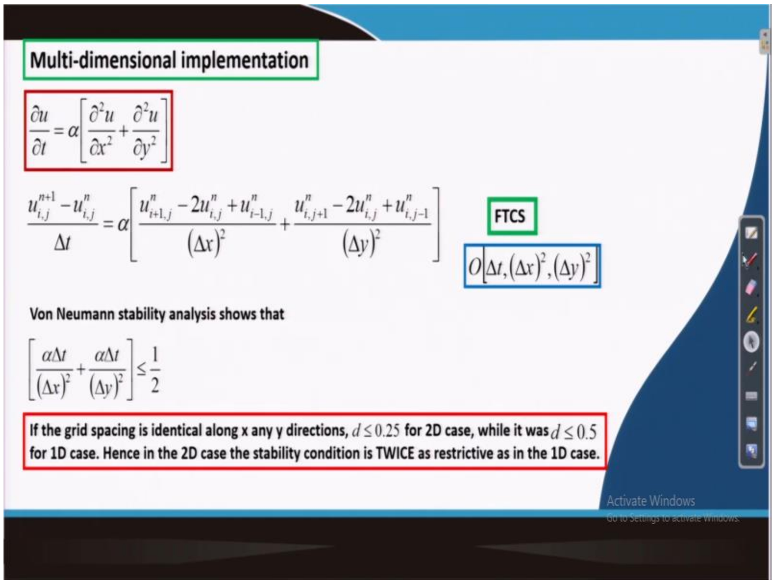

So, we now understand that applying the FTCS scheme to a multi-dimensional situation would become very costly and very, very time consuming exercise. So, what could be a better way of extending. The solution procedure for a multi-dimensional implementation, yes, of course, one way could be that we look for implicit means of solving multi-dimensional problems. So, we know by experience that implicit schemes do not have the stability issue. They are unconditionally stable. So, let us look at one such scheme.

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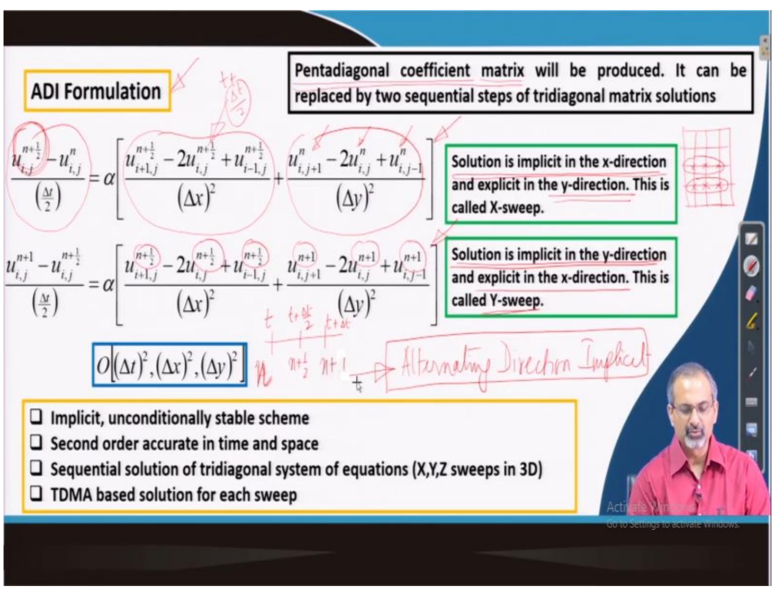

So, we have previously discussed about the ADI formulation just to remember what that means. So, it essentially means alternating direction implicit. So, of course, that means that it certainly is an implicit scheme with no stability restrictions. But, we are also talking about alternating direction that means, we would have to look at the different directions in an alternating manner not at the same time.

And this procedure had been discussed earlier in the context of elliptic equations, but we are again going to look at it. Now; in the context of parabolic partial differential equation. So, we talked about the concept of x sweeps and y sweeps. So, let us try to look at the different sweeps that we are talking about and the discrete form of the equation which is applicable for the different sweeps.

So, we realize that if we were to write down the equations in an implicit form as a single step process that would end up producing a pentadiagonal coefficient matrix, which is rather difficult to handle in the sense of generating a solution for the unknowns through a matrix inversion. And therefore, we would like to convert the problem into sequential steps of tridiagonal matrix solutions and in order to do that.

The alternating direction implicit scheme is a very, very effective scheme. So, it essentially splits the directions and tries to solve the problem the multi-dimensional problem in the form of separate sweeps along the respective coordinate directions. So, if we look at the x sweep, we say that the solution is implicit in the x direction. And explicit in the y direction so, the first thing that we notice is on the left hand side the way.

We discretize the temporal term is very similar to what we did for the Crank-Nicolson scheme when we look at the right hand side term, now that it is a multi-dimensional situation. We are discretizing the x derivative at the n plus half time step that means, the augmented time step. So, we have the nth time step here, the n plus half time step here n plus half times step here. So, this corresponds to t, this 2 t + delta t by 2 and this to t + delta t.

So, here these values are corresponding to t + delta t by 2, which is the n plus half time step. So, naturally we have implicitness along the x direction, because, if you look at the y derivative, they are all at the nth time step which are knowns and therefore, they would not introduce any implicitness into the solution. So, here we talk about solving a tridiagonal problem, where the unknowns has spread across the x direction.

So, we take up the entire solution domain and sweep it by rows. So, if the solution domain looks like this, and you have grids of this kind, so, you are solving the use simultaneously in a given row and then moving on to the next row and so, on. So, this would mean the x sweeps and then once you are done with it. The solution is available at the n plus half time step. With that you proceed for the Y sweep where the solution is implicit in the y direction and explicit in the x direction.

So, explicit in the x direction means that the solution is at n plus half time step where the values are all known now, and now, you are moving the values to n + 1th time step by sweeping along the y directions and taking them to the n + 1th time step. So, if we were to sketch the things, then it would show up in a form like this that we have the spatial grid available at different temporal levels.

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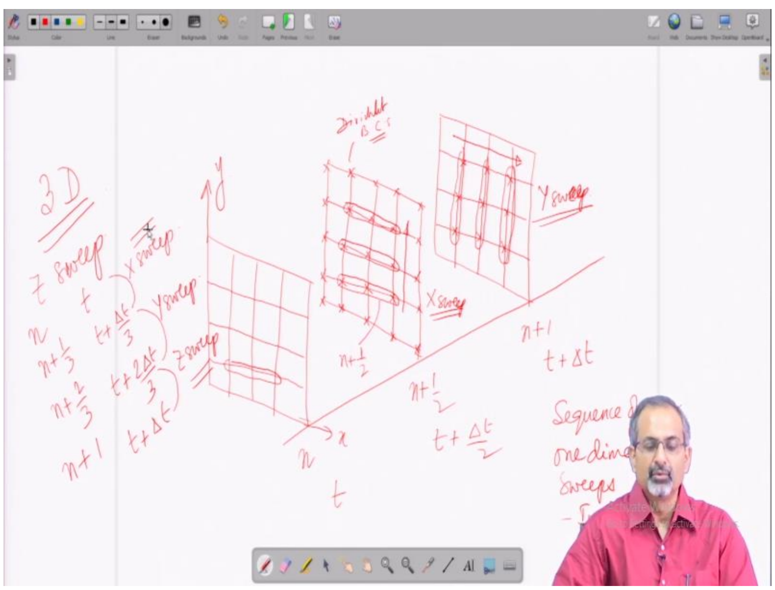

So, say, let us say this is n, this is n plus half, this is n + 1 which means this is corresponding to t this is t + delta t by 2. This is t + delta t and then what we have inside the domain is a set of gridlines disposed along x and y direction. Let us call this x, this as y. So, we were talking about x sweeps. So, when we do the x sweeps. We are not here at n but rather we are actually doing the x sweeps when we take it from n to n plus half.

We take up the unknowns as they are disposed in rows and we take them to the n + 1 at a level. Of course, remember that these points are never taken to the next level, if they are posed as boundary points of the Dirichlet **(()) (24:29).** So, they are not updated. Now, once you have all the solutions at n plus one half, then the next sweeps are done along ys that means the points disposed along columns would be updated one by one.

So, here you may be doing it row by row going from the lowest to the uppermost row. Here, you may be doing from the left and to the right and covering the entire domain in the form of columns. So, this would give you the x sweep and this would give you the y sweep. And as a combination, you end up solving the multi-dimensional problem in the form of a sequence of one dimensional sweeps. So, we have implicitness in one direction only during a given sweep.

If we were to solve the same problem in a three dimensional sense, then there would be some minor changes here, you would additionally have z sweep but before we do that. We have to first split the time levels from n, we go to n plus one third, then to n plus two third, and then finally to n + 1. So, that would mean t, t + delta t by 3, t + 2 delta t by 3 and t + delta t. This would involve x sweep, y sweep and z sweep.

Again, there is nothing fixed about how you sequence them. It may be done in different manners over different time steps. So, if you may follow up a x y z sequence by a y z x, a z x, y and so on. So, that would eliminate any kind of bias in the solution also, which may be coming due to the particular sequence of sweeps that we are using.

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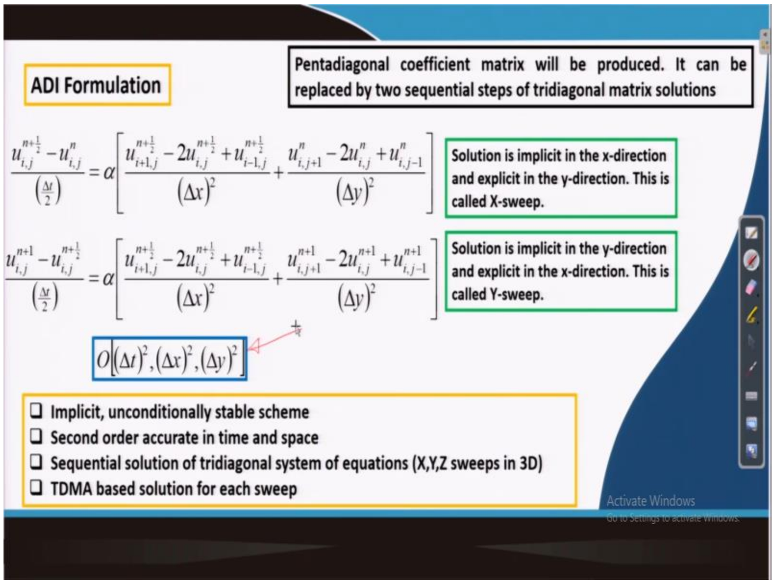

To recall once more, we have discussed now, an implicit formulation which is unconditionally stable. Interestingly, like the Crank-Nicolson scheme, it has second order accuracy in time and space, more importantly multi-dimensional space. And the problem is solved essentially as a sequence of solution of tridiagonal system of equations using the different sweeps in multi-dimensional space which can be two dimensional or even three dimensional.

And the TDMA algorithm is used individually for each such sweep. So, with this we complete our discussion on parabolic partial differential equations. Thank you