Introduction to CFD
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Lecture - 23
Numerical Solution of Unsteady Heat Conduction (Parabolic PDE) (continued)

We continue our discussion on parabolic partial differential equations.

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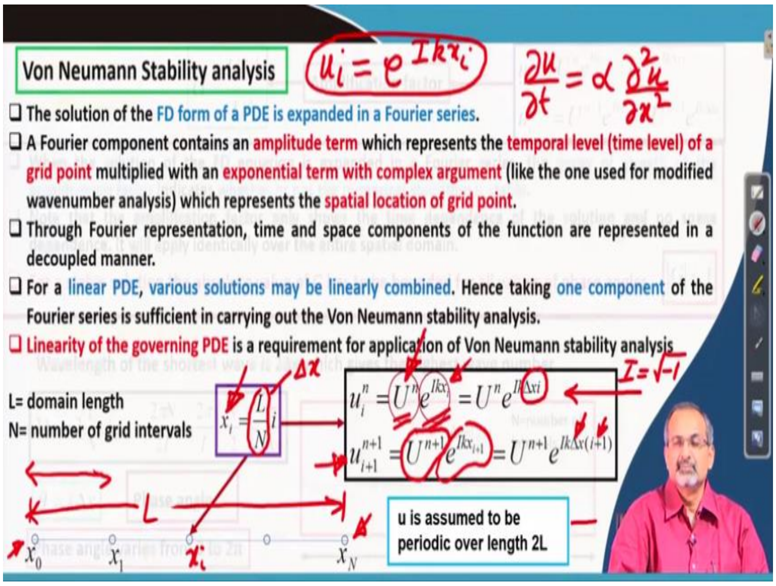

In the previous lecture, we had started discussing about Von Neumann stability analysis, in order to test a finite difference scheme to ensure that it is stable, before we use it in order to solve parabolic partial differential equations.

So, we had already noted, 1 or 2 initial points that we would take the finite difference form of the partial differential equation and expand it in a Fourier series and each of the Fourier components would comprise of an amplitude term and an exponential term with a complex argument which takes care of the spatial location of a great point. Additionally, since we are decoupling space and time.

When, we represent it through the Fourier components in this manner. Therefore, each component of space and the time would remain in a decoupled manner, in the Fourier representation. Now, an important aspect is that when we are looking at linear partial differential equations, the various solutions can be linearly combined to generate newer ones and therefore, taking just one component of the Fourier series should be sufficient in carrying out the Von Neumann stability analysis.

So, instead of taking a large number of terms or an infinite number of them, we just pick one and try to study the stability behavior, using that 1 term and this simplifies the whole process in a very big manner. Now, linearity of the governing partial differential equation is of course a general requirement for carrying out the Von Neumann stability analysis. So, let us say that if you are not handling a linear partial differential equation.

Then how would you be able to apply the stability analysis at all. So, the answer to that would be that you have to find a way to linearize the nonlinear partial differential equation in some manner before you can apply the Von Neumann stability analysis. But, if you do, then you should be in a position to apply it, at least locally. This is one aspect. The other aspect is that it is not possible to include the effect of the boundary conditions when you do the Von Neumann stability analysis.

So, when you are looking at the finite difference scheme in its discrete, the finite difference representation of the partial differential equation, then what you are essentially doing is you are looking at its form as the **(()) (03:15)** way it would be applied for the inner grid points, the internal domain grid points. So they do not show the influence of the boundary grid points. So, keeping all these points in mind.

Let us look at the mathematical representation. Now, so that we can apply Von Neumann stability analysis and try to understand that whether the forward time central space discretization which we learned about last time is actually a stable discretization and is it unconditionally stable, or it is stable under certain conditions. That is something that should come out from the stability analysis.

Now for doing the stability analysis, some of the ideas that we already came across when doing the modified wave number approach, also come up over here. So when we represent the problem. Here we have one spatial dimension of the problem because we are handling the governing partial differential equation of this form. So it only has one space dimension that is x.

And let the domain have a length L along the x direction, starting from the **(()) (04:33)** leftmost point x n ought to the rightmost point x N and so, these are the geometrical locations or coordinates of those points. And you could easily convert them into grid coordinates, because a certain point i would have a geometrical coordinate of x i, which is given by this ratio L by N times i.

So, L is the length and of course, N is the number of intervals that you have through the discretization. That means the number of grid spacing's that you have in the domain. So what does that give you as a ratio it essentially gives you, delta x, which is the spacing between 2 adjacent grid points. So, in order to get x i, what you essentially do is, you multiply, delta x by the grid location i.

So this is something that we also did while discussing about the modified wave number approach. Now coming to the way we represent u, i, n. So you remember, then when we did the modified wave number approach. We just had the spatial dimension to be tackled and therefore, we had written it as this. This was the way we did it over there, because only space had to be tackled. Here you have both space, as well as time to be tackled.

So what we have done is we have defined the so called amplitude term associated with small u, the amplitude, we call as capital U and it has a superscript n. So it essentially means the amplitude of U at the nth time interval or time level and that is multiplied by the spatial part of the function, which remains the same as what you did in the modified wave number approach. So it is e to the power of capital I k x i, but of course capital I stands for under root -1.

Now, we remember that x i could be represented as a product of delta x into i. So, this is how we represent u, i, n. Similarly, you could represent the u at other spatial with points and other time levels. For example, u i+1 n+1 would be represented by the amplitude term, which will be corresponding to the n+1th time step and the exponential term, which would now have a grid coordinate of i+1 and that would be multiplied by the grid spacing delta x.

So, that is how u i+1 n+1 would be represented periodic over **(()) (07:53)** the length L.

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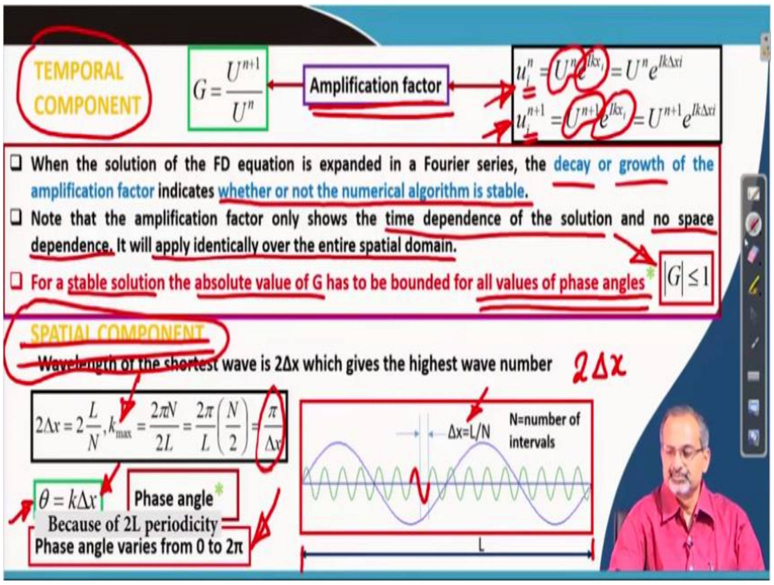

We have emphasized on the amplitude, corresponding to the temporal part of the time interval part and the exponential term, which has a connection with space. So, if we look at these as temporal components and spatial components from the Fourier term. Then what is going to be very significant for us is to track what is called as the amplification factor as far as the temporal component is concerned.

So, we are looking at the amplitude term at 2 consecutive time intervals. One is the nth time interval. The other is the n+1th time interval or step. And we are trying to take a ratio between the amplitudes for these 2 consecutive time steps that defines what is called as an amplification factor. So what we are trying to figure out is that whether the value of capital U is growing with time, or decaying with time, or remaining the same with time through a ratio of this kind. So, as we do that we remember of course, the expressions for u i n and u i n+1.

So, as you remember the spatial terms would remain the same, because they are corresponding to the same grid point i. But, the amplitude terms get changed, because they are the ones which carry information about the time. Now, it is important for us to track this, because when we are solving the partial differential equation through a finite difference equation, we are expanding the finite difference equation in a Fourier series.

When we try to understand its stability behavior, through Von Neumann stability analysis and we are trying to figure out whether this amplification factor is decaying or growing with time and that would essentially ensure whether the numerical algorithm actually behaves in a stable manner, or not because if there is a growth in this capital U which stands for the amplitude of the solution.

Then the solutions will become larger and larger in amplitude and then become extremely large and then become unbounded in course of time as you **(()) (10:37)** the solution and that is a recipe for instability and we would like to avoid that at any means. When, we are trying to discretize the partial differential equation through a finite difference approximation. So, the finite difference approximation should be such that the amplification factor remains bounded.

And it satisfies a very important property that it is modules should be less than or equal to 1. Another important point that we would like to mention over here is that when we look at the stability aspect, we are looking at amplification factor which is just carrying information about 2 consecutive times it carries no explicit information about the spatial distribution of the solution.

Though, it comes into the expression for U i n+1. Sorry, U n+1 by U n ratio implicitly. But, there is no explicit representation of the spatial distribution. So we are, as though, just worrying about the time dependence of the solution and there is no space dependence explicitly seen in the amplification factor. So, whatever is the amplification factor, we can say that it applies identically over the entire spatial domain.

It is again important to note that for a stable solution, the absolute value of G has to be bounded for all values of phase angles. Now, we recall from our previous discussion on modified wave numbers, the definition of phase angle. So we said that when we multiply the wave number by the grid spacing. So if you call the wave number as K as we did before and you multiply it with the grid spacing delta x, then you get theta, which we call as the phase angle and we know that.

So, as you do that the phase angle varies from 0 to 2pi. Now, if you look at the distribution of a function over the length L. Then we discussed earlier that you could have very large wavelength waves, as well as very short wavelengths accommodated within this grid. So, you can see the grid spacing delta x as given by L by N, where N is the number of intervals and for the shortest wave, you will have a wavelength equal to 2 times delta x.

So if you track the shortest wave that can be captured, it will be looking like this. So that covers a length of 2 times delta x. So, that would correspond to the highest wave number, which we can define as k max and it can be worked out that will be equal to pi by delta x, so that is the highest frequency wave that you can accommodate spatially within this domain. So, this is the spatial component of the Fourier term.

So, the spatial component part of it is already known to us from our previous discussions. However, we were not aware about the temporal component so with the discussion on temporal component as well as spatial component combined, we now have a better understanding as to how each one of the terms like u, i, n would be represented in a Fourier sense so that we can now launch into the activity of performing the Von Neumann stability analysis.

And for doing that our candidate scheme is the FTCS scheme. So, we will try to write down the discretized form of the FTCS scheme once more when we carry out the Von Neumann stability analysis.

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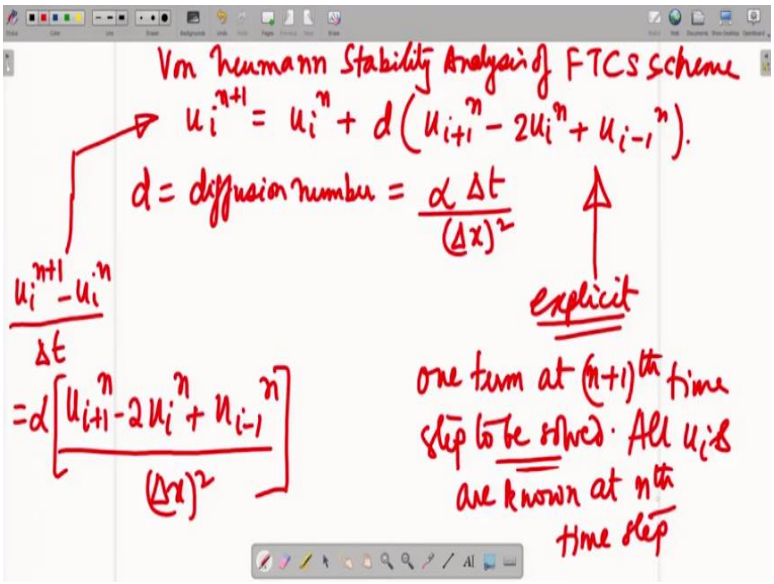

We remember that this could be written as. Now here, d stands for diffusion number, which is nothing but a combination of the diffusion coefficient alpha and the time as well as space intervals. So, it is alpha delta t by delta x square. So, this equation, of course, is obtained from the original form of the discretization, which is this by rearranging the terms. So, we write down the original form once again.

So that it is convenient for us to connect between the 2 forms. So, for the stability analysis, this is a more convenient form as we will see soon. Again, we recall that this is an explicit scheme, because you have only 1 term at the n+1th time in step to be solved and we know that all u i's are known at nth time step. Now, let us try to use the Fourier representation for each and every term.

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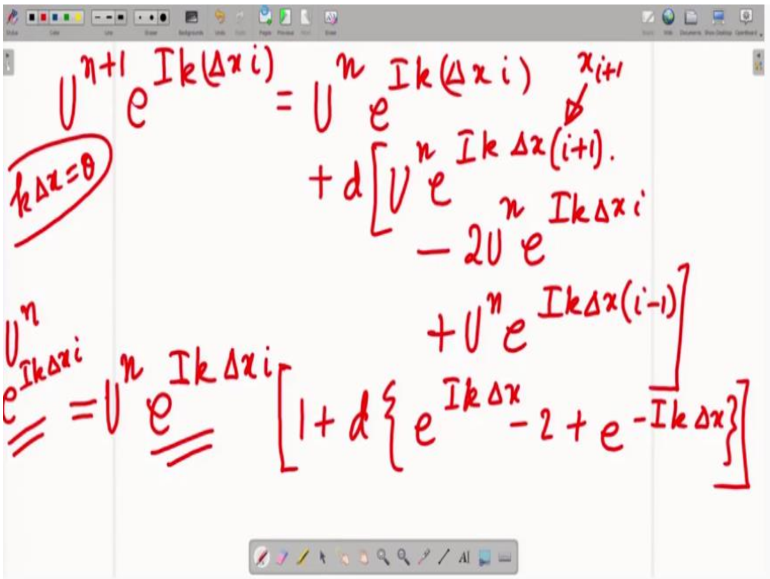

Initially, we will write the k times delta x times, as they occur in the basic definition. Later on, we will replace k times delta x by theta that is the phase angle. So, if you write it down term by term. So, this is nothing but x i+1. This completes the substitution. Now, we can have some common terms. Let us try to write them, which can be put outside the brackets.

So, we notice that on the left hand side also, you have this e to the power of I k delta x i. So, this and this term can cancel out. So let us put it in a simplified manner like this.

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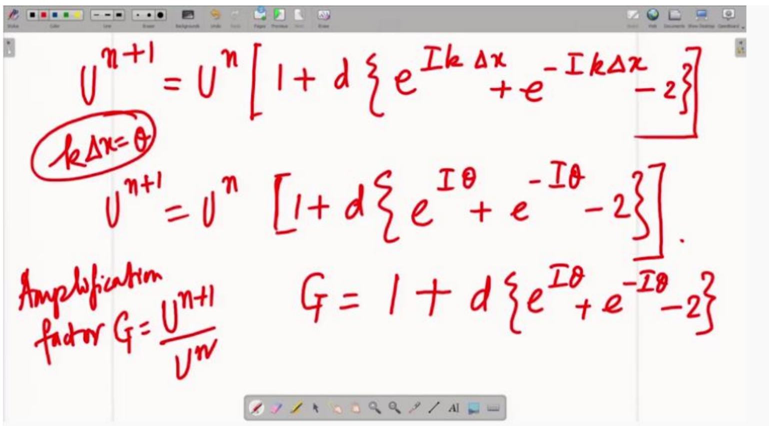

And now, additionally, let us put it in terms of the phase angle in the next step. So, as we mentioned earlier that we can replace k times delta x by theta, which is the phase angle. So then you have this equation and we defined amplification factor, G. So, let us bring that here is equal to U n+1 by U n. So you have G is equal to 1 + d into e to the power of I theta + e to the power of -I theta -2, this is the expression. Now, we recall the Euler formula.

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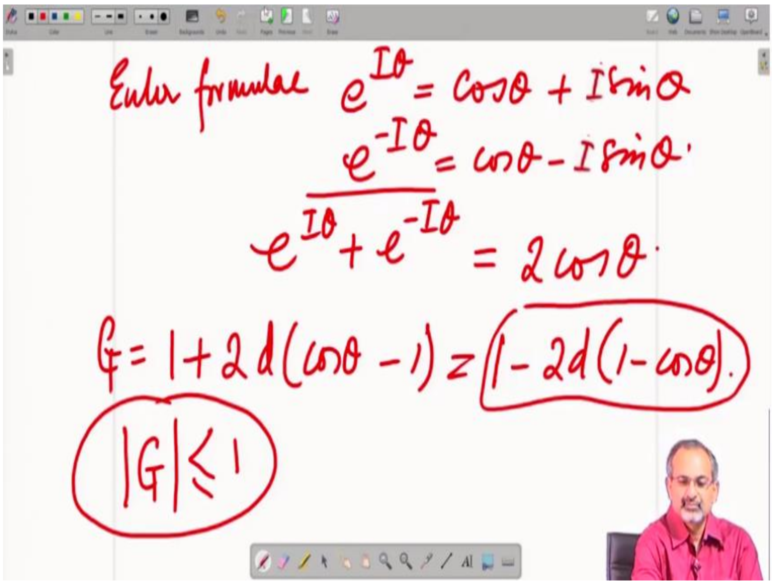

So, we have e to the power of I theta is cos theta + I sine theta and then e to the power of -I theta is cos theta - I sine theta. So, using these 2 equations, you can sum them up so that you have an expression for e to the power of I theta + e to the power of - I theta because that is what figures in the G expression and that will give you 2 cos theta.

So finally, if you look at the expression for G, what you have is G is equal to 1 + 2d cos theta -1, let us write it as 1 - 2d into 1 - cos theta. So our idea was to check whether mod of G remains bounded. Through this condition that it does not ever exceed 1. So we need to check whether this expression satisfies that condition. So, in order to do that what we really need to check is that whether it satisfies this inequality.

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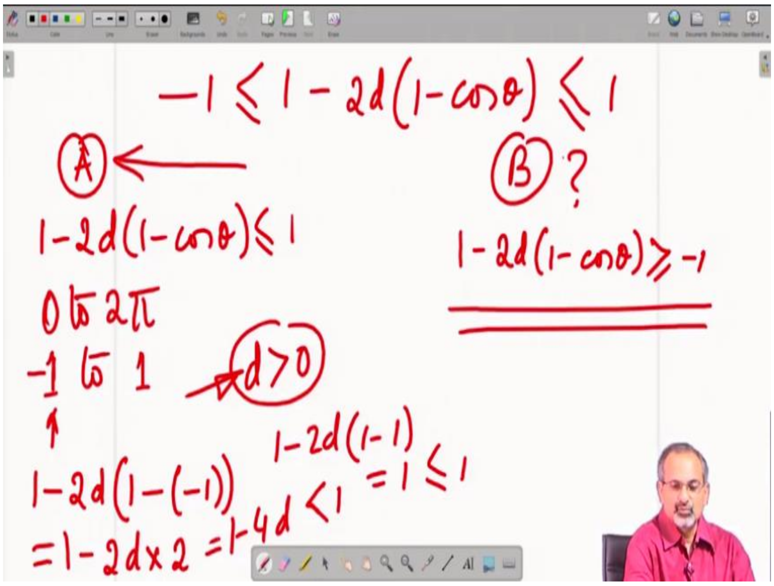

If you look at it as 2 parts, then one part is that let us say, part A and part B. So, let us put part A as 1 - 2d into 1 - cos theta. We need to check whether it is less than equal to 1. The other part is that whether 1 - 2d into 1 - cos theta is greater than or equal to -1. These are the 2 things we need to check. Now, if you think about all possible values that cos theta can attain between 0 to pi, rather to 0 to 2pi.

It will vary between - 1 to 1. Now, if you substitute these values, you will always find that let us say if you substitute the value of -1 for cos theta, what do you have you have 1 - 2d, into 1 - -1 which is 1 - 2d into 2 and that is, 1 - 4d and we have to recall that d > 0. It is a diffusion number which is essentially dependent on the diffusion coefficient which is always a positive number times delta t, which is a positive number divided by delta x square which is also a positive number.

Therefore, the this, the value of d would always be greater than 0 and therefore 1 - 4d will always be less than 1. If you look at the other extreme that is when cos theta is equal to 1. Then you have 1 - 2d into 1 - 1, which essentially gives you 1 and therefore it also satisfies the inequality, it is less than equal to 1, which means there is absolutely no issue in satisfying the criteria A, so, one of inequalities.

So, we have to now question, the second inequality, whether it is satisfied for any value of theta, like the criteria A did. So, for checking that let us do some calculations.

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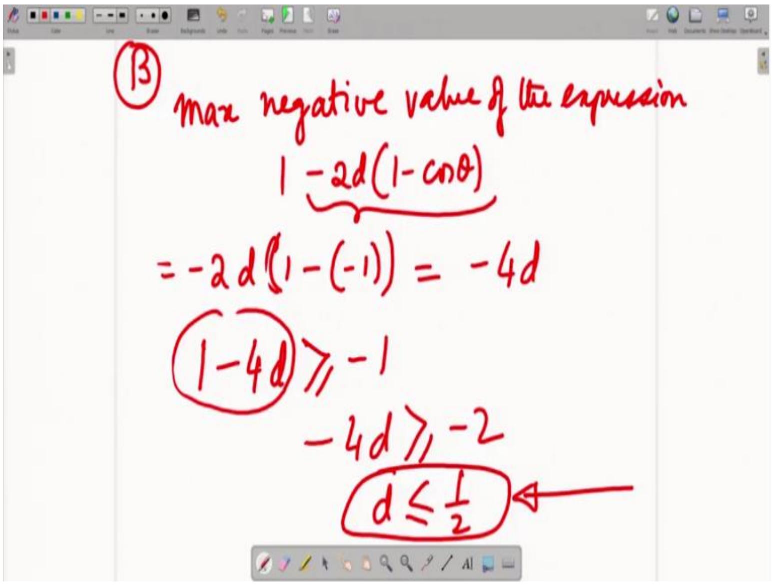

So for the inequality B that has tried to put in the values possible values of cos theta and try to figure out what how it works. Now the main issue is, you have to ask the question, what is the max negative value of the expression, 1 - 2d into 1 - cos theta and that would mean that we are trying to figure out what is the maximum negative value of this bracketed term. Now, the maximum negative value can occur when you have a -1 coming in from cos theta, because that could make this term equal to -4d.

So that can have the maximum negating effect on 1 and our condition is that 1 - 4d has to be greater than -1. That means, at no point can d be that big that this expression becomes lower than -1, or smaller than -1. So, what is the limit then? The limit is this and therefore d should be less than equal to half, so that is the condition that we are getting from the inequality B.

So what does this mean, if we just think about the expression for d, it is this and that should be less than or equal to half.

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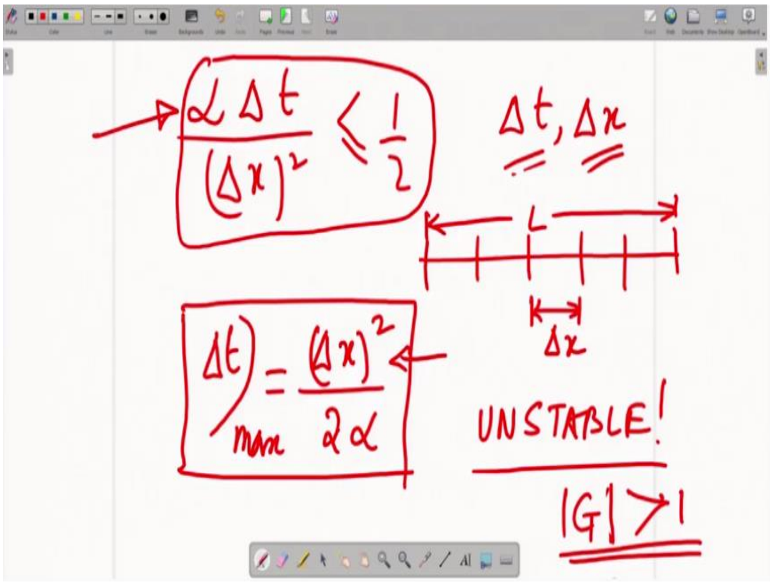

It means that for a given problem, the diffusion coefficient will have a certain fixed value. We cannot control that. But what we can control is our choice of delta t and delta x. So, we have to choose the delta t and delta x rationally so that you always end up satisfying this condition. Now, you can boil it down further to saying that well, I have a certain length and I have decided to have such and such grid spacing.

So in that case, you are also facing fixing up the value of delta x. So what is left for you to fix up then is delta t, which would be fixed up by using a condition looking like this. So, your delta t should always be less than equal to delta x square by 2 times delta, 2 times alpha. That would be the maximum delta t, so the t max could then be defined as equal to this. If you exceed it, then the solution will become unstable.

And that is something that we need to always avoid. So, how can you make out that the solution has gone unstable? Of course, if you plug back this value of delta t which is exceeding this value, then you will obviously be able to get mod G greater than 1 and that will make the solution unstable. Now how an unstable solution would look like, is something that we will discuss later. So we close our discussion here today.

We look at some more results on parabolic partial differential equation and especially interesting results on stability in the next lecture. Thank you.