Introduction to CFD
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Module - 1
Lecture – 2
Governing Equations of Fluid Flow

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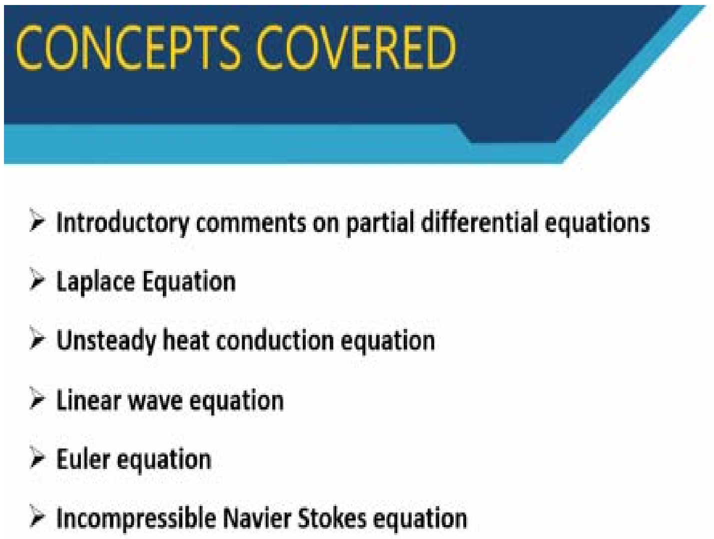

In this lecture, we are going to look at the governing equations of fluid flow. We will start from very simple model equations and try to build them to more complex levels. So, this lecture broadly looks at some introductory comments on partial differential equations. Then we go over to some simple model equations where we look at Laplace equation to begin with. Then we look at unsteady heat conduction equation followed by linear wave equation.

These are individual partial differential equations, which I mentioned. After this, we are going to talk about more complex equations, which actually form a system like Euler equation and incompressible Navier Stokes equation. We have to keep in mind that this is not an all encompassing list. There are many more equations in the domain of fluid dynamics. At this point, we do not need to bother much about that. This would give you some introductory exposure to some of the important equations.

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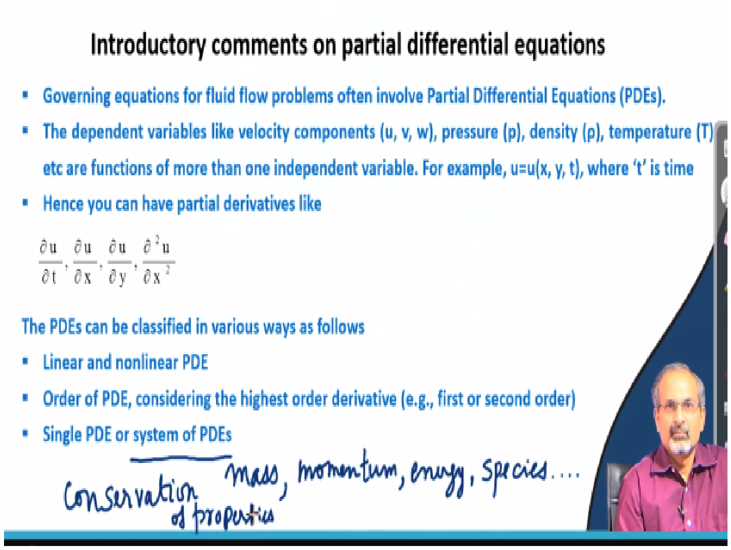

When we talk about governing equations for fluid flow problems, we very often see that they comprise of partial differential equations. These partial differential equations could be single equations for simple problems while they could be a system of partial differential equations for more complex problems, Henceforth, we may very often be referring to partial differential equations as PDEs, which is a very commonly followed abbreviation.

To have a brief description of what exactly we mean by partial differentials, we can have various dependent variables in a flow field like velocity components. So, if you are looking at it as a Cartesian system where you have the x, y, z directions you could have velocity components defined along these respective 3 directions. So, then you talk about the u, v, w components of velocity.

You could have pressure, you could have density, temperature and other parameters which could be of interest when you are solving a fluid flow problem and most often we find that these variables are dependent on a number of independent variables. For example, when I talk about the x component of velocity u, I may be finding that u is actually a function of a number of independent variables, which are x, y, t and could be more.

When do I have u as a function of x and y, here it means that we are interested in solving a flow problem which involves two spatial directions x and y, which are orthogonal to each other. Additionally, there seems to be a dependence of the u on time. So, that is why the u component of velocity now becomes a function of 3 independent variables. So, remember that x, y, t are independent variables while u, v, w, p, rho, T and others which you can think of which carry information about the flow field are dependent variables.

We are always interested in a given CFD exercise to calculate the dependent variables as a function of the independent variables. Now, let us look at what are the different derivatives that we can calculate once we know that u is a function of x, y and t. So, now you know that there could be a derivative of u with respect to the time, derivative of u with respect to the x direction, y direction and remember that these 3 are all first order derivatives because you are applying the derivative operator only once.

$$\frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$$

However, they are not ordinary derivatives. They are partial derivatives. Therefore, we indicate them with the del sign and not the d sign. So, you can take a derivative of u with respect to t, you can do it with respect to x, with respect to y and possibly more independent variables if they exist. Also, you can apply the different derivative operators once or multiple times. When you operate it twice one in succession of the other, then you get a second derivative.

So, how did you get the second derivative? You first took a derivative of u with respect to x and you got $\frac{\partial u}{\partial x}$, you again took a derivative with respect to x of $\frac{\partial u}{\partial x}$ and therefore you ended up getting $\frac{\partial^2 u}{\partial x^2}$. Likewise, this may be happening for other dependent variables also. So, when you are solving a flow problem where all these are your dependent variables, you can imagine that there could be numerous derivatives, both spatial as well as time derivatives, which can exist.

It is very important for us to understand that these derivatives will be important for us because in fluid dynamics world we end up handling partial differential equations, which are comprised of these derivatives. Having said that, we can now understand that once those derivatives make their way into the equations, the equations may be classified in different possible ways.

One very important way by which we classify partial differential equations is written here, where we say that a partial differential equation could be linear or nonlinear, we really need to understand what is meant by that.

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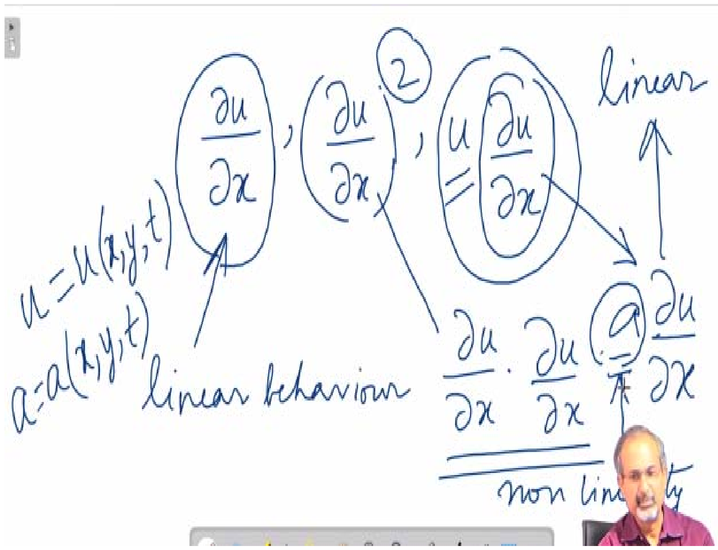

Here you find a number of derivatives written. This is something that you have already looked at. The first derivative of u with respect to x, $\frac{\partial u}{\partial x}$, it is a partial derivative, a first order partial derivative. So, this derivative has a linear behavior because of the following reasons. One, that it is not raised to a power other than 1. When can this happen? It can happen when you are actually multiplying this derivative with itself $\frac{\partial u}{\partial x}.\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2$. This introduces non linearity.

This is one possible way by which non linearity can creep into a fluid dynamic equation. What are the other reasons? The other reason or at least one of the other reasons could be a term which looks like this where you have the dependent variable being multiplied with its own derivative.

$$u\frac{\partial u}{\partial x}$$

This is another major source of non linearity. You will very often come across with this kind of non linearity in fluid dynamics equations.

Notice that if this same term was written in a manner like this, where you have a constant coefficient 'a' associated with the first derivative, that would not lead to non-linearity.

$$a\frac{\partial u}{\partial x}$$

So, this is a linear term in a partial differential equation because this constant has no connection with the dependent variable. However, there is no guarantee that the coefficient would always be a constant. It could depend on independent variables.

For example, in a problem where we stated that u is a function of x, y and t, i.e., u=u(x,y,t), we could say that even this coefficient could be a function of x, y and t, i.e., a=a(x,y,t). In that case, you will find that this constant will become dependent on the spatial location and time, but nevertheless that would not be sufficient reason for considering that term to be a nonlinear term, only thing is that then that coefficient becomes a variable.

These are very important concepts, which we have to understand before we actually try to classify a partial differential equation as a linear or a nonlinear one and then it makes a lot of difference. That is why we are spending some time on this aspect to understand it very deeply. Getting back to our slide where we were looking at the points based on which we will classify partial differential equations.

We now understand that how the derivative terms would look like in a linear or a non linear partial differential equation. Apart from this classification, there is more to the classification. We go to the next point where we talk about the order of the PDE. Here, we look at the derivatives which constitute a partial differential equation and we see what is the highest order of derivative which exists in that equation?

Is it first order or is it second order or it is even higher? Now, what do we mean by order when we talk about a partial differential equation. What we understand is that if we have a term which looks like $\frac{\partial u}{\partial x}$, then the order of the derivative is 1. However, when you have a second derivative, $\frac{\partial^2 u}{\partial x^2}$, then the order of the term is 2. As we said that there could be even higher orders than 2, but not very often seen in fluid dynamics.

Going to the next point, we can have a single partial differential equation or system of partial differential equations when we look at a flow problem. So, we need to understand that how the system or a single partial differential equation emerges for a given flow problem, by and large depends on the complexity of the flow problem. For relatively simple flow problems or associated transport phenomena, which could also include heat transfer for example, we could have a single partial differential equation, which can take care of modeling the phenomena. However, for including more complex phenomena, most often we need system of partial differential equations and often these systems take care of conservation of certain properties. It can involve conservation of mass, conservation of momentum, conservation of energy and many more things. It could be conservation of certain species. It depends on how complex a system you are trying to model. So, what we understood was that system of partial differential equations often take care of conservation of certain properties and this is very important.

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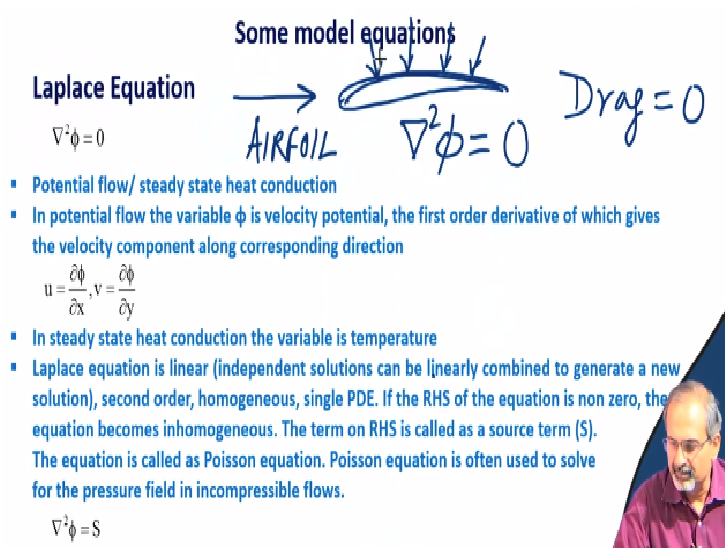

We are now looking at some simple model equations. This we must appreciate is just relative simplicity. We do not really mean to say that Laplace equation, which is a comparatively simple model equation is actually very simple, but yes it is simple in comparison with a much more complex system of equations like say Navier Stokes equations. Now, what is the statement that we have over here when we look at Laplace equation?

So, we have a so called Laplacian operator which is indicated by the del square. So, it is a partial differential operator which is operating on a scalar which we have indicated as phi.

$$\nabla^2\phi = 0$$

So, let us try to write a little bit more on what this operator actually means.

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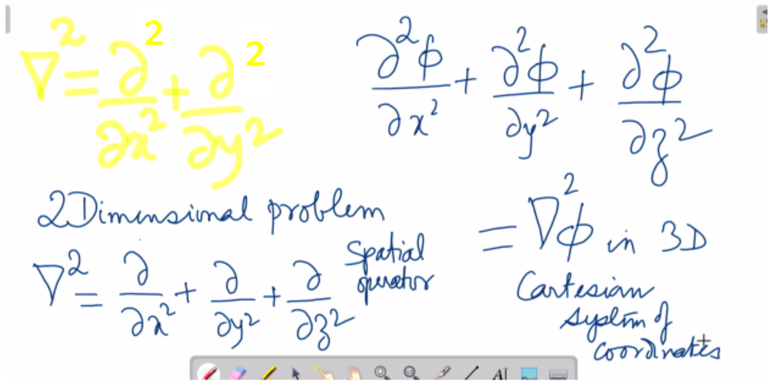

What we do over here is write the Laplacian equation or the operator for a 2-dimensional problem.

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

So, what would it look like when we have a 3-dimensional problem? You can just imagine that one more dimension will get added to it.

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

So, what do we have over here? We have second order partial derivatives, which form this operator and every time we include one term means we are just catering to another direction in space, but it is a purely spatial operator.

It has nothing of time involved in it. So, when we apply the Laplacian operator to a scalar which we have indicated as phi, then how will the terms look like? So this would give you Laplacian phi, $\nabla^2\phi$, in 3 dimensions for of course a Cartesian system of coordinates.

$$\nabla^2\phi = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

We have available expressions for the Laplacian operator in other orthogonal systems as well. You may be interested to solve a problem, which is not very convenient to handle say in a Cartesian system.

In such a situation, you may not like to use this form of the Laplacian operator at all. You may like to use the Laplacian operator for a Cartesian system only for problems which fit well into a Cartesian coordinate system, while there may be problems which can be better tackled using spherical coordinate system or cylindrical polar coordinate systems. In such cases, we have to invoke the corresponding expression for the Laplacian operator.

This is important for us to remember because the governing partial differential equations look different based on what kind of coordinate systems we are using. So, when we apply the general form, we just write Laplacian phi = 0, $\nabla^2\phi = 0$, then we are not explicitly stating what kind of coordinate system we are using. So, only when we write it down in detail, it becomes very explicit that whether we are using a Cartesian system or a spherical system or say a cylindrical polar coordinate system.

In fact, in CFD to tackle realistic problems, we may not always be able to fit in these kinds of so called orthogonal coordinate systems all the time. You may very often be handling non-orthogonal coordinates. In such cases, you may be at best able to approximate the operator, but not write it down exactly. Coming back to what we had here on the slide, we see that Laplace equation is used to study potential flow.

So, we are talking about a kind of flow which is devoid of viscosity, is incompressible and very importantly irrotational. So, these are the very important features of potential flow. We often say potential flow is ideal flow because we have idealized it through these significant assumptions or simplifications. However, on doing this, we get a remarkable simplification of the governing equations and that is why the entire flow problem can now be encapsulated into a single governing equation.

If you look at the general form of the equations from where we start simplifying, they are by no means a single partial differential equation, they are actually a system of partial differential equations, but when we apply these assumptions, then the system finally emerges as a single partial differential equation. So, whenever we do CFD, we have to keep track of assumptions that we are making down the line because a lot of the simulation results basically depend on the capability of the model equation.

Again, we have to be extremely careful in setting the proper initial and boundary conditions appropriate with the governing equation so that we generate a sensible solution out of it. Also, we need to interpret the results with care. We cannot expect a model to do beyond its own capability. For example, if I am using Laplace equation to solve flow past an airfoil section, which is looking like the wing section that we talked about earlier in the pictures, which looks more like this.

So there could be a flow coming and meeting this airfoil section and then if you are solving such a flow using Laplace equation, you need to understand that you will not be able to simulate any kind of viscous effect on the airfoil surface, which means the airfoil will not feel any drag. Drag force is felt by a body when immersed in a viscous flow which resists its movement through that flow.

So, when you are running very quickly, you can feel the wind blowing against your face, you can feel a stronger wind, when you are driving a scooter or a motorbike. You can imagine what kind of winds are faced by the windscreens of aircrafts. So, when the wind impinges on these surfaces, very often in the real situation, it ends up dragging that surface behind, resisting its motion.

Now, if you are solving a problem across the airfoil where you have used Laplacian phi = 0, $\nabla^2\phi = 0$, as your governing equation, it fails to predict drag. So, what it predicts is 0 drag, which is not the reality. However, we have to understand that that comes from the limitation of the model equation because that is rooted in the assumptions. This is a very important consideration when we are doing computational fluid dynamics.

We have to define for ourselves what exactly we are looking for. So, you may remember that there were 5 important steps in a CFD exercise. The first step was problem definition. In the problem definition, you have to define yourself that when you solve flow past this airfoil section, whether you are interested to calculate the drag or not. If you are interested in calculating drag, then certainly Laplace equation will not help.

You have to perhaps look at more complex equations, at least the boundary layer equation. In certain cases, you may actually have to look at more complex equations, which could be the Navier Stokes equations. However, Laplace equation can do a remarkable job in predicting what pressure is felt by this surface.

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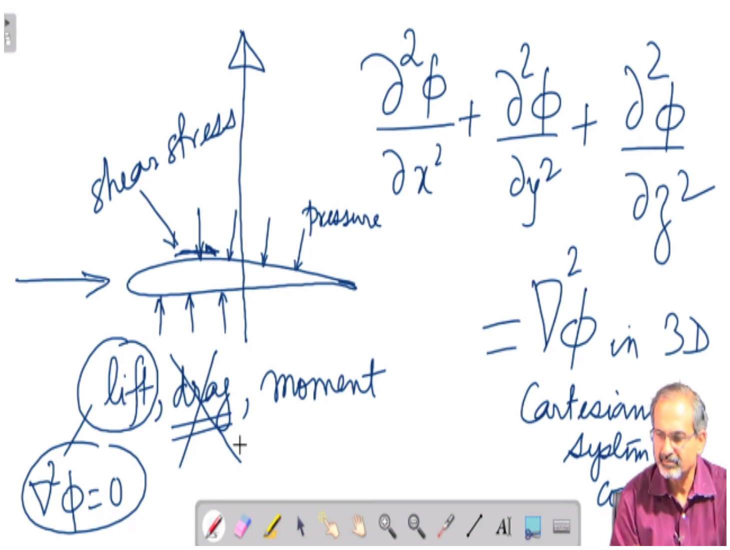

We will try to draw that airfoil section briefly and try to understand what kind of forces may actually be coming on to that surface when it faces the flow. So, as the flow moves past the surface, pressure acts on the surface all over. The pressure tends to point towards the surface. Additionally, there is shear stress which is acting on the surface and it remains tangential to the surface at every point.

So, you have pressure on one hand and shear stress on another. A combination of pressure and shear stress would be able to give you a realistic estimate of lift, drag and pitching moment which act on this airfoil section. So, when you are interested in lift alone, which is the force which tends to keep this airfoil section afloat when it is flying in air, that can be very well predicted with Laplace equation.

So, this is one of the main motivations of using Laplace equation to solve potential flow. However, as we already discussed that it would not be able to predict drag because of the accompanying limitations.

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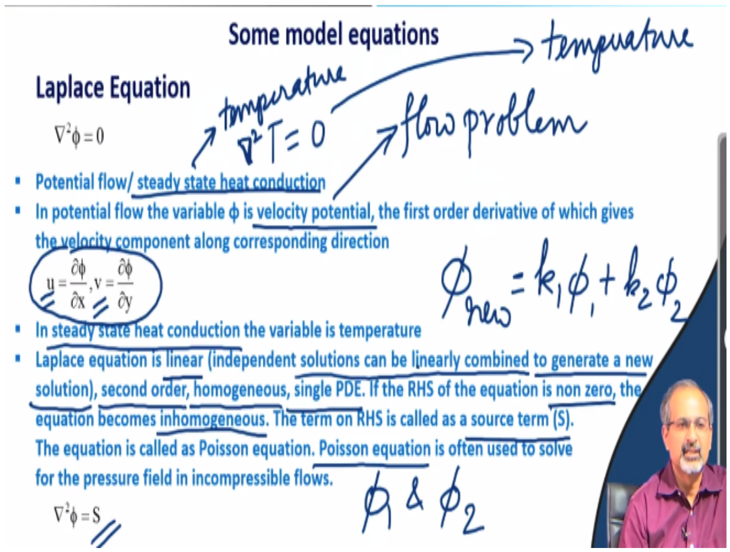

We can use Laplace equation even to solve the steady state heat conduction problem. So, instead of phi acting as a flow variable which we call as the velocity potential when solving potential flow past the airfoil section, when solving the steady state heat conduction, phi represents temperature distribution in a certain region.

Then the equation actually looks like this Laplacian T equal to zero, $\nabla^2 T = 0$, where T stands for temperature. So, if we are interested to find the steady state temperature distribution in a certain region, then Laplace equation can be used to do that. So, we find at least two very important applications of Laplace equation in fluid dynamic and thermal or heat conduction problems.

Just to make the story more complete, what does the velocity potential mean over here? You find that different velocity components which we talked about earlier when we were talking about partial derivatives are mentioned here; i.e., u, v, and they are connected with phi. So, apparently if you take partial derivative of the velocity potential which we represent as phi along different Cartesian directions, we retrieve the velocity in those directions by doing that. Therefore, if you are able to solve for phi from Laplace equation, you should be able to solve for the velocity field.

In steady state heat conduction equation when you use temperature as the dependent variable, then the approach is very similar, but then you do not have to do this additional step of connecting the velocity potential with the velocity components by taking derivatives of phi, you get the solution straightaway. Now, looking at the behavior of Laplace equation, there are a few very important things we need to notice. The first thing is that Laplace equation is a linear partial differential equation. Why is it?

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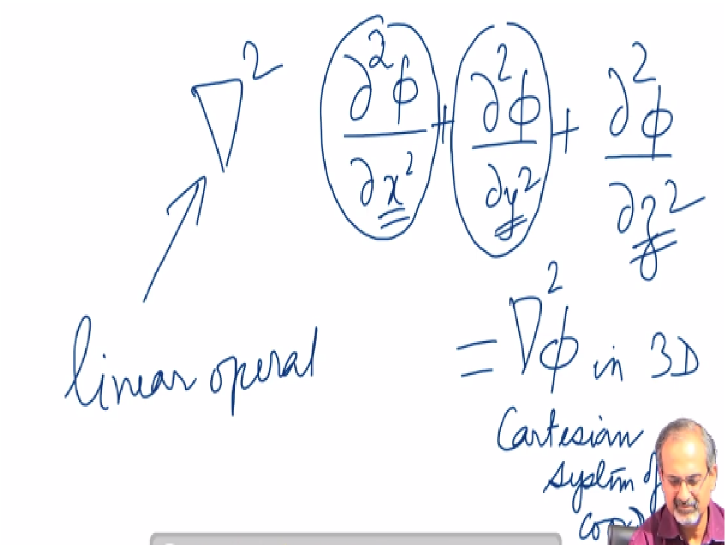

So if we go back and try to do a little work to understand what we meant by linear, you will recall that if we are not raising the power or the index of a derivative to anything other than 1, then the term remains linear. Also, if we are not associating the dependent variable with its own derivative, then also there is no reason why non linearity will come. So, in terms like these, you will not have any reason for non-linearity.

Second order spatial derivatives constitute the Laplacian operator. Each one of them produce linear terms and therefore the Laplacian operator is a linear operator. There is a very important characteristic associated with linear operators. If you have 2 independent solutions of the Laplace equation which you write as ϕ_1 and ϕ_2, then ϕ_1 and ϕ_2 would independently satisfy Laplace equation.

But most interestingly, you may be able to generate another new ϕ, which is a linear combination of ϕ_1 and ϕ_2, which also satisfies Laplace equation. So, you can have a new ϕ which is a linear combination of ϕ_1 and ϕ_2, $\phi_{new} = k_1\phi_1 + k_2\phi_2$, where k_1 and k_2 are constants. Then this ϕ_{new} can also satisfy Laplace equation. You can try doing this yourself during spare time and see for yourself whether this works or not, but I can assure you that this works.

This is a very important feature of linear partial differential operators or linear partial differential equations. So, solutions can be linearly combined to generate newer solutions. This is a very important feature. Additionally, Laplace equation has second order derivatives and therefore it qualifies as a second order partial differential equation. It equals to 0 on the right hand side and that is what makes it homogeneous and it is a single partial differential equation which solves the flow problem.

Therefore, we do not need to solve more than one equation to generate the flow solution here, which makes us a bit lucky of course. If the right hand side of the equation in Laplace equation is nonzero, then we have it in its inhomogeneous form and then we call it as the Poisson equation. If we have terms on the right hand side, which can be indicated as 'S', then Laplacian phi equal to S, i.e., $\nabla^2\phi = S$. Such non zero terms on the right hand side of the equation are called as source terms.

Let us say we are trying to solve a steady state heat conduction problem in a domain which looks as follows.

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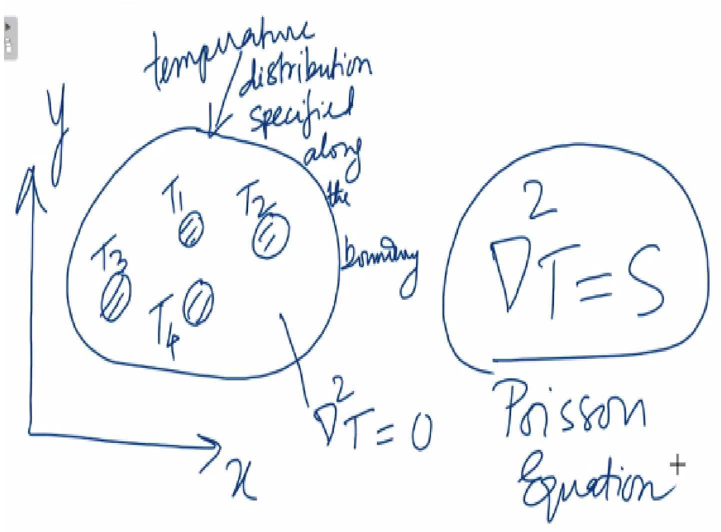

So, we have a planar domain as above, which can be defined typically in the x, y coordinate system. Laplace equation would be solved to obtain the temperature distribution within this domain. The temperature distribution has to be specified along the boundary of the domain. So, until and unless you define the temperature all around the domain boundary, you cannot expect Laplace equation to solve for the steady state temperature distribution inside the domain.

Having said that, one may pose a problem like this that is it only going to be the temperature distribution all over the domain boundary which will help us do realistic problems? We may like to define certain points within the domain which are acting like hotspots. So, you may have certain internal regions where temperatures are different from that specified at the domain boundary.

These temperatures could be higher or even lower than what exists on the boundary. The main point is that if you do want to put such sources or sinks of temperature, then you need to end up using this form of the equation $\nabla^2 T = S$. Solving Laplace equation $\nabla^2 T = 0$ will not answer this question. So, you essentially solve for Poisson equation and this is the equation with the source term on the right hand side.

Poisson equation incidentally is very often used in solving the pressure field in incompressible flows. Incompressible flows usually occur at lower speeds, though that is not a very formal definition of it.

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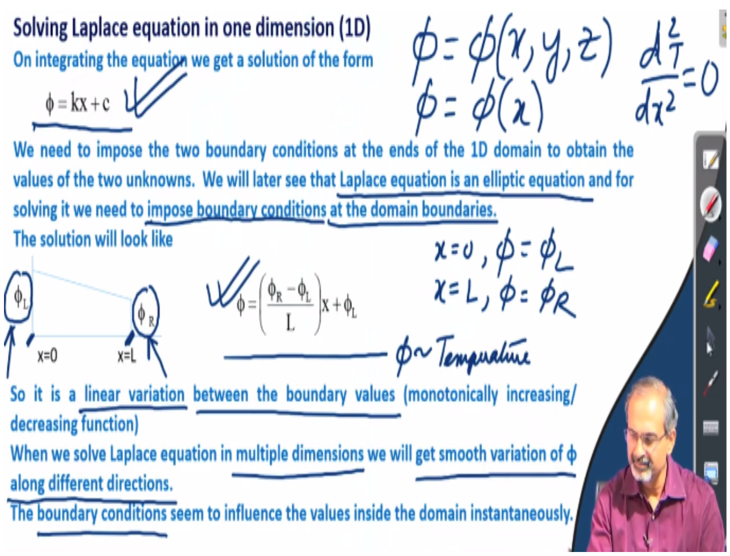

Let us do a very simple problem here. Let us try solving the Laplace equation in one dimension. Now, incidentally, if you reduce the Laplace equation to one dimension, then essentially the partial differential nature of the equation will be lost, because it will then become an ordinary differential equation. In general phi could have been a function of x, y and z, $\phi = \phi(x, y, z)$, but we are simplifying the problem and saying that phi can only be a function of x, $\phi = \phi(x)$. Dependence of ϕ on a single independent variable, x, reduces the PDE to an ODE.

The number of independent variables involved in the problem reduces to one and you can take derivatives only with respect to x, which means the problem now boils down to involving ordinary derivatives. So, ordinary derivatives can give you ordinary differential equations and in the given perspective Laplace equation would now look like $\frac{d^2T}{dx^2} = 0$ instead of $\frac{\partial^2 T}{\partial x^2} = 0$.

Now, this is a much simpler equation to integrate as you can understand. If you integrate it twice, you will get a solution looking like this $\phi = kx + c$. However, we do not know what the value of k and c are. Of course, these can come by suitably imposing boundary conditions, which we specify at the domain boundaries. A few minutes back, we were already saying that in order to solve Laplace equation, we must always specify boundary conditions all over the domain.

So, there it was a planar domain, but here it is going to be just a one dimensional domain because we have reduced the problem to only dependence along x. So, let us say you have a domain which is defined from x = 0 to x = L along the x direction and you have specified the boundary conditions of ϕ as ϕ_L and ϕ_R taking L and the R as the left and right ends of the domain.

Now if you have a solution looking like this $\phi = kx + c$ acting on this domain, then you can very easily find out what the values of k and c are when you impose these boundary conditions in the above equation. As you can understand that at x = 0 you will have $\phi = \phi_L$ and at x = L, $\phi = \phi_R$. So, if you impose these conditions in the above equation you will be able to find out k and c and then finally the variation of phi along the length of the domain.

So, if ϕ is representing temperature, then this kind of a problem is connected to steady state heat conduction in one dimension and what we find over here is that this problem has a linear variation of temperature over the length of the domain. Again, we make a comment over here which we will discuss at length in a later lecture that Laplace equation incidentally is an elliptic partial differential equation.

This is one more way by which we can classify a partial differential equation, however we have not discussed much about it. Only thing that we know is we have to impose suitable boundary conditions at the domain boundaries in order to come up with a solution in such a case. The solution shows a linear variation between the boundary values and it will never show an overshoot or undershoot compared to the boundary values.

It will always remain bounded between the minimum and the maximum boundary values. In this case, ϕ_R and ϕ_L. When we solve Laplace equation in multiple dimensions, even there we will get a smooth variation of ϕ, but that will be along multiple directions, which will generate surfaces or volumes. We understand that boundary conditions have a profound influence on how the solution emerges. This is a characteristic behavior of partial differential equation.

Interestingly in Laplace equation, there is no time dimension which means the solution does not take time to evolve to a steady state as though we have an instant answer, an instant distribution available to us. Now in the heat transfer problem, we say that we are interested only to look at the steady state temperature distribution and therefore it is obvious that we are not going through those times when the temperature is actually changing.

We are only looking at the final snapshot, the equilibrium temperature distribution. If you look at the physics of the problem and go deep in to the material, probably there will be a lot of molecular collisions going on to take care of the heat transfer from one end of the plate to another till it reaches that steady state distribution. However, we are not interested to find any details regarding those transient states or the intermediate states when the solution is actually unsteady that means it is changing with time.

We are only interested to find out about the equilibrium state. How does it work when it is a flow problem like we were talking about potential flow solution? Incidentally over there if you introduce any source of disturbance into a flow field, which could be an airfoil, which could be any other surface, which could also be a moving airfoil, then you have to first check whether the flow broadly satisfies the assumptions under which Laplace equation is valid. If found valid, then presence of a body or body movement would create pressure waves in the flow field, which travel enormously fast (in incompressible flows they travel at infinite speed) in all directions of the flow field and takes no time to adjust the flow properties. Therefore, you have a steady state solution, which does not need any time to emerge. So that is the trick which is available as far as the potential flow problem is concerned when we try modeling it using the Laplace equation. This is a very unique feature that we need to understand.

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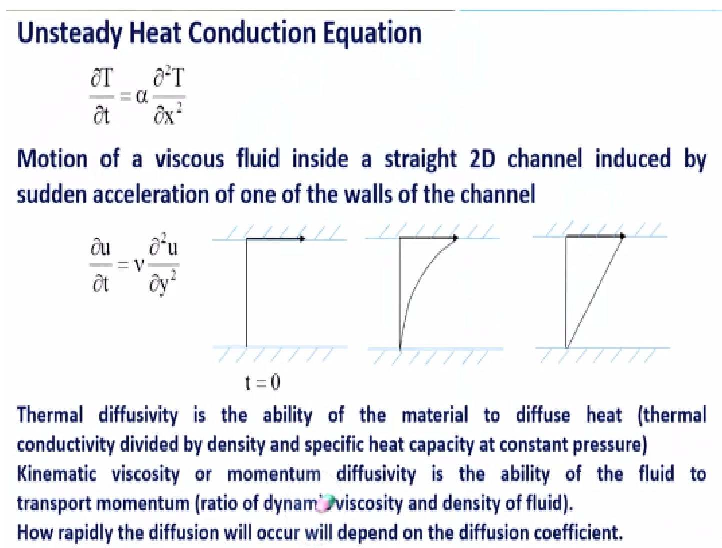

Another very important and interesting transport equation is called as the Unsteady heat conduction equation. One of the terms looks like the Laplace operator, which is seen on the right-hand side of the equation associated with a coefficient which we call as the thermal diffusivity. On the left hand side of the equation there is a time derivative.

This is the first model equation where we see dependence of the temperature on time. We did not look at any model equation where time would be important for us yet. It was not there in the Laplace equation, but it is now available in the unsteady heat conduction equation. What we are going to talk about over here is that how the temperature distribution in a one dimensional domain will start developing over time till it reaches steady state. This we will address in our next lecture.