Introduction to CFD
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Lecture - 11
Methods for Approximate Solution of PDEs (Contd.)

In the previous lecture we were talking about the Galerkin weighted residual technique. In today's lecture, we will continue with the weighted residual technique, but this time we will apply the weak formulation or the variational formulation.

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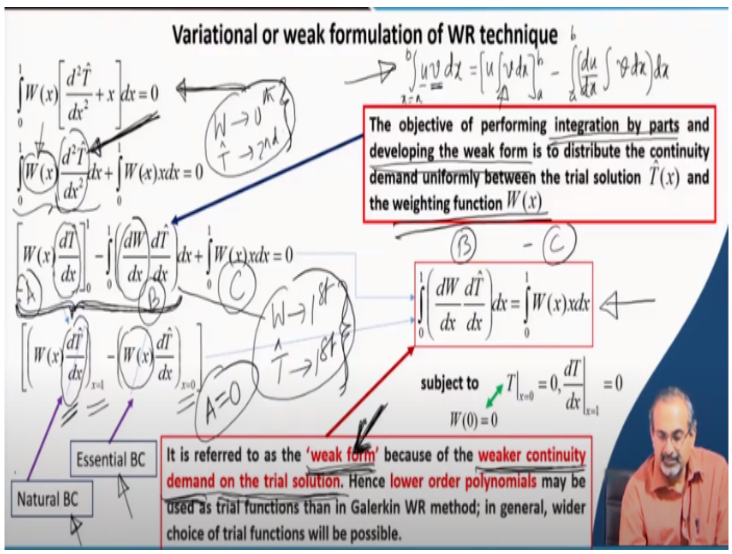

We discussed about this fact that in the Galerkin formulation, we use the differential equation with its highest order derivative. However, in the variational or weak formulation of the weighted residual technique, we apply the integration by parts in order to reduce the requirement of satisfaction of the highest order derivative in the trial function.

And therefore, the formulation looks a little different from what we saw in the Galerkin technique. So let us see, what are the things we do in order to propose the problem in the weak formulation form. Now before we do that, we need to understand what we actually mean by a weak form. For that you have to look at the bottom part of the slide where we have the word weak form mentioned again.

And it essentially means weaker continuity demand on the trial solution. Now we will understand more of this as we continue the formulation. The first step looks identical to what you have done earlier. And we are continuing with the same differential equation that we discussed in the previous lecture.

The Galerkin technique was talking about minimizing the weighted residual as stated in the first statement here. Now how does it change, when we have a weak formulation? What we do is, we first identify the terms where the highest order derivative is figuring. So we find that this is the term where the highest order derivative of the dependent variable is figuring.

Now we need to reduce it by some means and for doing that, we make use of the concept of integration by parts, which helps us to develop the weak form. So what are we doing here? We are essentially integrating this product $W(x)\dfrac{d^2\hat{T}}{dx^2}$. So as you do that, two terms will be yielded. You may recall having learnt about integration by parts, where you have two functions u and v of x.

And there are two limits let us say x is equal to a to b and then you write it as u integration v dx from a to b minus a to b du dx integration v dx and then whole dx. So this is how the integration by parts would be performed.

$$\int_{x=a}^{b} uv\,dx = \left[u\int v\,dx\right]_a^b - \int_a^b\left(\frac{du}{dx}\int v\,dx\right)dx$$

So in this case, which is your u and which is your v? As is evident, that you can reduce the order of the derivative for this term if you keep it as v, because you are actually getting v integrated over here.

The moment you integrate v, the order of the derivative will reduce once. If you integrate by parts once, the order of the derivative will reduce by one when you are integrating the function v. Therefore, we set the d^2T/dx^2 term as the v term. It is in the same sequence available here as we needed. So once you do that, now if you compare with this form, you can understand that that is exactly what you have over here.

$$\left[W(x)\frac{d\hat{T}}{dx}\right]_0^1 - \int_0^1\left(\frac{dW}{dx}\frac{d\hat{T}}{dx}\right)dx$$

So you have the two terms and interestingly the highest order derivative has got reduced. But additionally a derivative of the other term $\frac{dW}{dx}$ has figured. Now what does that mean? You had a zeroeth order derivative of W and a second order derivative of the T term in the original formulation. So W was zeroeth order and $\frac{d^2\hat{T}}{dx^2}$ was second order in the original form.

Now what do you have? W has now been raised from zero to first order and T has been reduced from second to first order. So if you sum the orders over here, the sum of the orders remains the same. But they have just got redistributed. That means the continuity demand on the trial solution has reduced now. Earlier it was that it would have to be continuous up to the second derivative.

Now the requirement is that it has to be continuous up to the first derivative, but simultaneously the weighting function now also has to be continuous for the first derivative. So what have you done? You have essentially distributed the continuity demand uniformly between the trial solution and the weighting function. This is a very, very important aspect of the weak formulation.

In the process, we have actually reached the desired goal of having it in a weaker form. That means weaker continuity demand on the trial solution. This is the cornerstone of the variational formulation of the weighted residual technique. Incidentally, the weighted residual technique also figures in the finite volume technique where the weightage function is set to unity.

Here, we continue to use the same technique as we used before. That is we will rely on the trial solution to obtain the weighting functions. But we have to additionally get their derivatives now in the present formulation. Now looking at the problem that we are solving, let us see how this first term works out after you have put the limits. So we will call these three terms as A, B, C.

A: $\left[W(x)\frac{d\hat{T}}{dx}\right]_0^1$; B: $\int_0^1\left(\frac{dW}{dx}\frac{d\hat{T}}{dx}\right)dx$; C: $\int_0^1 W(x)xdx$

We are essentially looking at the term A and we are applying the limits 0 to 1. As you do it, you find that the first term has a derivative which goes to 0 at x = 1 by the boundary condition at x = 1 while the second term which is for the x = 0 has w going to 0 at that end because T has to go to 0 at that end by the boundary condition.

So in the process essentially A has become 0 for the given problem, which has simplified the problem statement altogether for us, of course. Now you are only left with the B and the C terms. We sent the C to the other side and then essentially write it in this form.

$$\int_0^1\left(\frac{dW}{dx}\frac{d\hat{T}}{dx}\right)dx = \int_0^1 W(x)xdx$$

When you are imposing boundary conditions of this kind in the A term that we were looking at, when you have these derivative terms, we generally call them as the natural boundary conditions.

When we set the values of the dependent variables or other independent variables to 0, then we call them as essential boundary conditions. We will try to work out the problem now through the weak formulation. As we said that for us, only the B and C terms remain.

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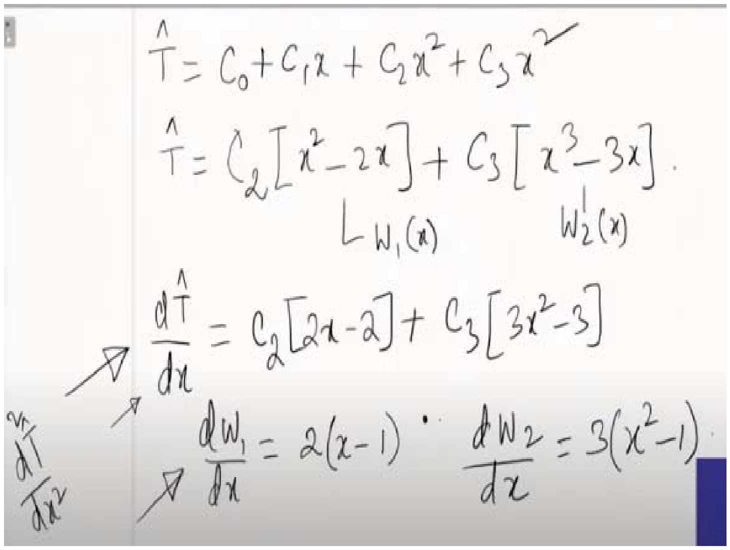

And let us try to recall that what trial function we had used earlier, we will continue with the same one. And we recall last time having worked out this form, we call this as W_1 the other one is W_2. And this time remember that you would only need the first derivative of $\hat{T}$ and additionally, you will also need the first derivative of the weighting functions.

So this is the first derivative of $\hat{T}$ and let us see what are the first derivatives of the weighting functions. So this is how you calculate the derivatives. Now remember that in the Galerkin technique, we had to actually go for the second order derivative and therefore, continuity at second order would have to be ensured. Now, the continuity requirement has reduced here. But additionally we need the derivatives of the weighting functions.

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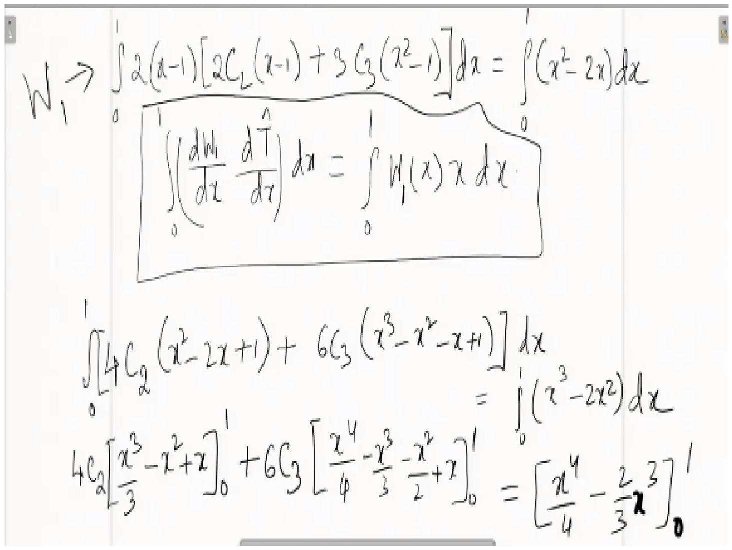

Now let us write down the residual minimization equations in this case. So you can understand that what we have done essentially is we have written this statement. Here what we have done is we have written it for W_1. Similarly, next time we will do it for W_2. So this was the form which we already showed applicable for our problem and this is what we are working out for W_1.

So if we do the integration, what will we get? Let us do the calculation. Now you have to integrate it term by term and then what you will get would look like this. So you are done with the left hand side and then on the right hand side you will get, this is what you will get. Now you need to put the limits.

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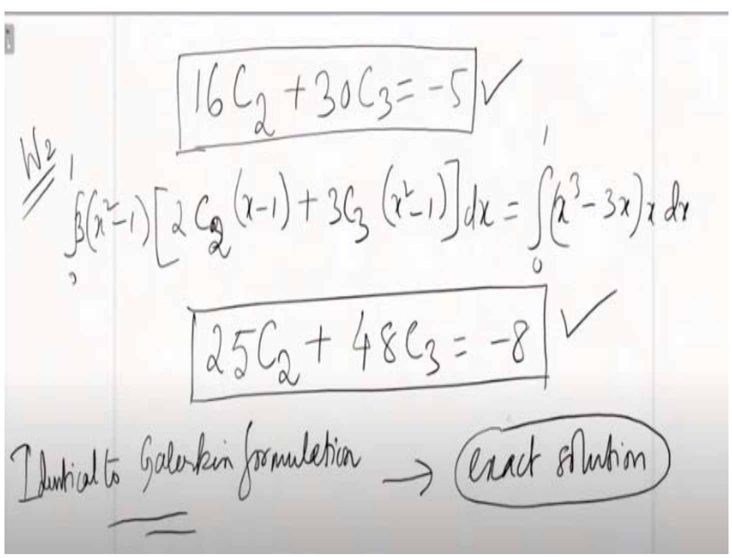

Once you put the limits, what you will get is an equation involving C_2 and C_3 which looks like, now let us look at minimizing based on W_2. So like we did in the previous case, we put down the first derivative values sorry this should be C_2 on the left hand side in product form and the right hand side contains only the residual sorry the weighting function times x. So this is based on W_2.

Now we similarly integrate term by term and we will find that what comes out of this exercise is an equation in C_2 and C_3 which looks like this. Incidentally, these equations are identical to the one that we obtained through the Galerkin formulation. And therefore, this will also yield the exact solution like it yielded for the Galerkin formulation.

But interestingly here in this case, we had a weaker requirement of continuity on the trial solution. So it could be interesting to check other kinds of trial solutions and see how the solution emerges, with reduced requirements on continuity of the trial function. We will discuss about a few more points related to the approaches we learned.

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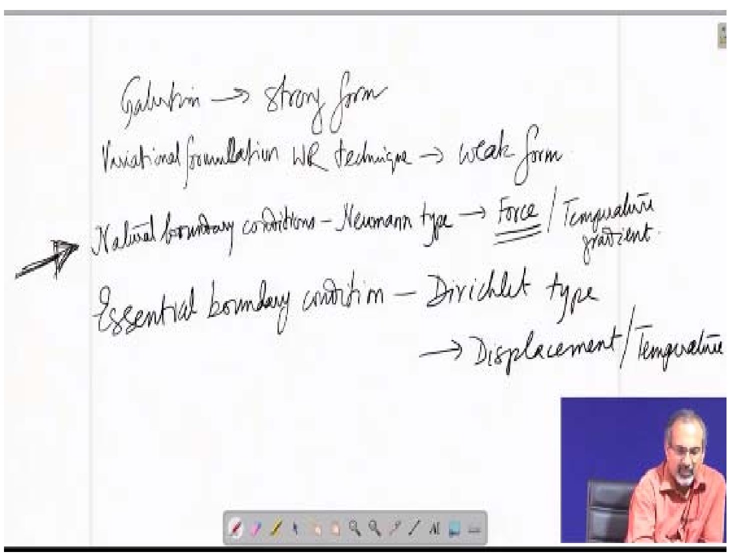

So we learnt that the Galerkin formulation uses the strong form while the variational formulation of the weighted residual technique uses the weak form. When we impose boundary conditions in the weak form, we often come across natural boundary condition which are usually the Neumann kind. So let us say if you are doing a problem in solid mechanics it may be a force term.

If you are talking about a thermal problem, it could be a temperature gradient term. The other kind of boundary condition could be the essential boundary condition, which is of the Dirichlet type. So in a solid mechanics problem you may like to specify displacement at a certain point. In a thermal problem you may be talking about specifying temperature.

So these are the kind of boundary conditions which you may encounter and the weak formulation can very conveniently incorporate the gradient kind of boundary conditions directly into the formulation.

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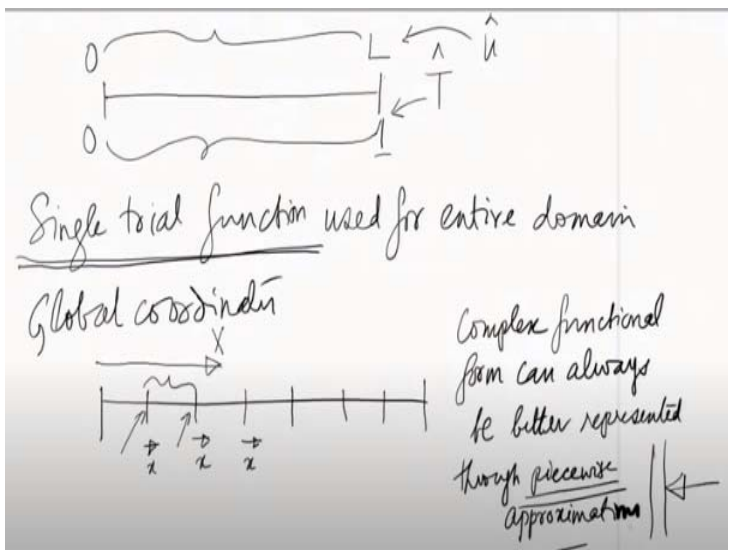

Additionally, what we need to discuss is that we talked about using a single function to span the entire length say 0 to 1 or 0 to L, whatever way you specify it for a certain problem. You remember that the first example that we solved, we had a length 0 to L. In the second it was 0 to 1. So whatever be it you had a single trial function.

$\hat{T}$ was addressing the problem of satisfaction of both boundary conditions and minimizing the residuals in the entire domain. So we were dealing with a single trial function used for the entire domain. Now we understand that there could be very complex variations possible for certain problems, in which case a single trial function may become increasingly inadequate to appropriately represent the functional variation.

In that case, it could be a very good idea to split the domain into a number of sub domains and then we essentially call these sub domains as the elements in a finite element technique. So you would then have a finite number of these elements spanning the entire domain. And then you would try to use similar kind of formalisms that we have developed for the entire domain piecewise on these sub domain elements.

For doing that, we have to do it carefully in the sense that when you do it for the entire domain, you may actually like to use global coordinates. While when you do it on subdomains, you define your entire domain and then you split it into a number of elements. Then this global coordinate system may be represented by say a capital X while the local coordinate systems would be the small x's.

And they could be defined for each and every element of this kind satisfying again the boundary conditions that you may be having at the ends of each one of those subdomains. Now the idea behind this would be that a complex functional form can always be better represented through piecewise approximations.

So this is one of the prime motivations of dividing a complex functional form into a number of sub domains and then trying to apply piecewise approximations which are like piecewise trial functions. And then if you are using the weak formulation, then you have lesser continuity constraints, which you need to satisfy anyway. So that applies even on those subdomains.

That means on the independent elements also you will have a weaker continuity demand. Therefore, you may end up using simpler trial functions with reduced continuity requirements which you have. So this is the advantage of handling a problem, which may come up with a complex solution using a finite element approach and then trying to apply the weak form and solving the problem.

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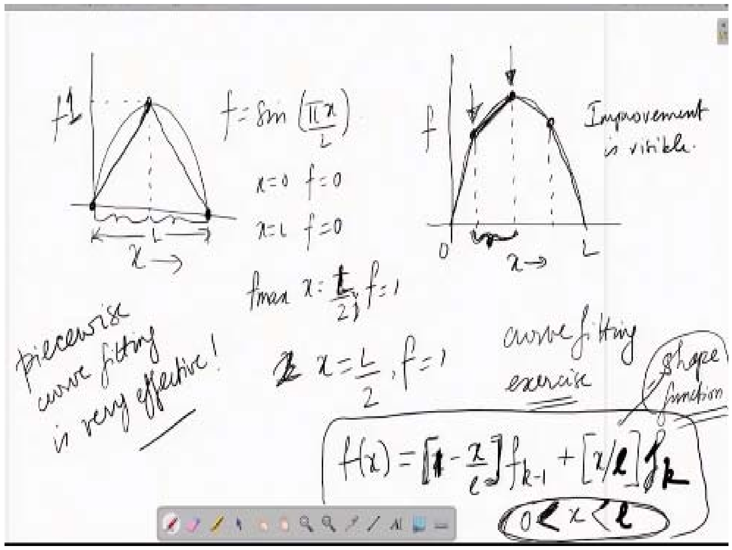

So if we really want to do graphically, one may say that if you are looking at a function like this. Let us say the function is like, so you know that at x = 0, f will be equal to 0. Again at x = L, f will be equal to 0. And the f_{max} will occur at x = π/2. That is where f will become equal to 1. It should be L/2. So that gives you sin(π/2). So let us write it a little more clearly, at x = L/2, f will be equal to 1.

Now if we were to divide this interval into two segments, then if you are trying to linearly interpolate, then you may come up with an approximation of this kind of the function. So this is like two elements to approximate the function. One may say that the moment you increase it from two elements to four elements you will see the improvement immediately. So it may now look like this.

And therefore, the improvement is visible. So like we were saying previously, that it is by and large, a curve fitting exercise and it seems that it could be very effective to do it piecewise. Without going into much details, we would say that in more complicated problems, we would not like to use a single trial function, but rather apply a large number of trial functions, separate trial functions for each one of the elements.

And then we may use interpolations of this form, which could be very useful. So remember that we were talking about using global and local coordinates. So if you assume that this x is applicable in the sense of a local coordinate, then if you are looking at a particular element, then there is a functional value at the left end, there is a functional value at the right end.

And you are essentially interpolating between those two functional values, so that you are able to exactly satisfy them at the ends. So that can be attempted using these kind of interpolation functions, which are often referred as shape functions in finite element nomenclature.

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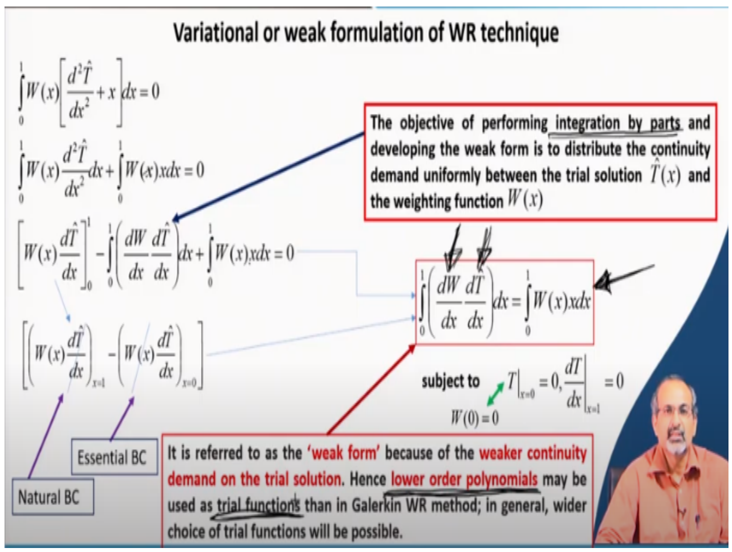

So we will quickly revisit the points that we saw when we were discussing about the weak formulation of the weighted residual technique. We said that this is a technique by means of which we can reduce the continuity demand on the trial solution, which is given by the $\hat{T}(x)$. And we essentially get it divided or distributed between the $\hat{T}(x)$ and the weighting function.

And for the given problem, we also saw that the problem statement became simpler when we applied the boundary conditions to one of the terms which came out of the integration by parts. And we noticed that here we were using the first derivative of T instead of the second derivative of T which we used in the Galerkin formulation. But additionally we had to use the first derivative of W.

And we have the opportunity of using lower order polynomials as potential trial functions in the weak formulation compared to what it was in the Galerkin formulation. So a wider choice of trial functions will be possible. And the weak formulation may also be more convenient in imposing the kind of natural boundary conditions in many problems.

So with this, we come to the end of the discussion on approximate solution of differential equations. We have by and large, concentrated more on ordinary differential equations because we looked at problems where the dependent variable was depending on one independent variable. However, the broad principles apply to partial differential equations as well.

And therefore, we would treat these techniques to be applicable for partial differential equations. And we will discuss many of those applications in the later lectures. Thank you.