Introduction to CFD
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Module - 2
Lecture – 10
Methods for Approximate Solutions of PDEs Continued

In this lecture, we are going to talk about another approximate method for solution of partial differential equations, which is called as the finite element method.

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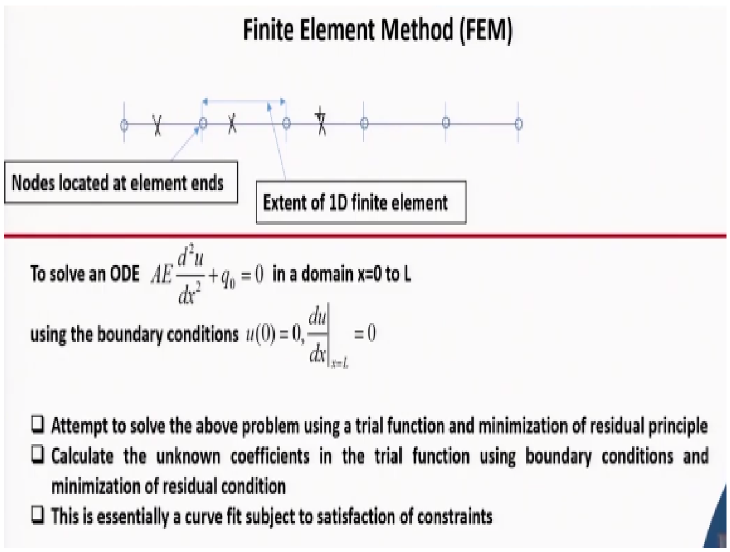

In the finite volume method which we have discussed earlier, we looked at a domain which was discretized into a number of finite segments which we called as finite volumes. In finite element method, the method in which we discretize the domain has a similar appearance, but this diagram shows that there are endpoints of these segments, which we have marked as nodes.

In the finite volume method, it was the central point which we were talking about as node. So, if you remember in finite volume situation we had nodes like this.

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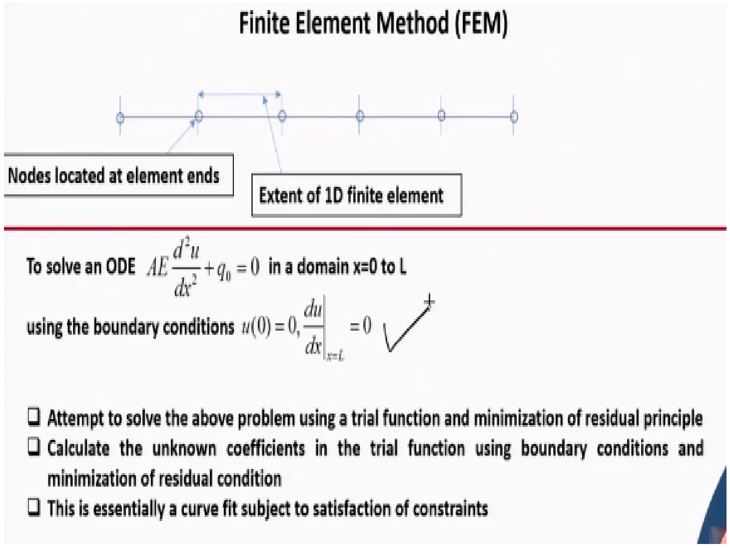

In finite element technique, we are talking about the endpoints of these segments as the nodes and these segments which we called as finite volumes are here called as finite elements. Now, we will actually look at solution of an ordinary differential equation here in order to understand some of the basic concepts of finite element method. So, we have an ODE given by AE d^2u/dx^2 + q0 = 0 which needs to be solved in a domain x = 0 to L.

Using the boundary conditions that u = 0 at x = 0 and du/dx = 0 at x = L. Incidentally, this equation is of importance in solid mechanics. Here, we attempt to solve this above problem using some trial functions and based on minimization of residual principle. As we do that, we will find that the trial function will be assumed to be a polynomial with certain unknown coefficients.

And these unknown coefficients would have to be evaluated based on the boundary conditions that you satisfy at the ends of the domain and also based on the residual minimization principle which we will talk about within a few minutes. You will realize at the end of an example that this is essentially a curve fitting exercise, and this curve fitting is done based on satisfaction of these above constraints that we talked about.

That is on one hand the boundary conditions because it is a boundary value problem and on the other you try to minimize the residuals all over the domain as you work out these coefficients.

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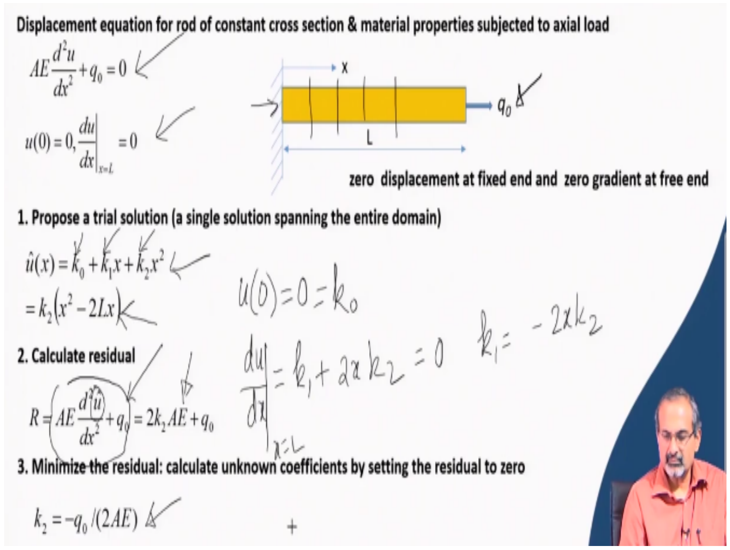

Let us look at the physical significance of the problem. So, you have an element on which you are applying a load q0. The element is a one dimensional one of length L and you are essentially modeling the displacement of different sections of the element from the fixed support through this governing equation. Now, we have assumed a constant cross section as well as constant material property which surfaces through the Young's modulus.

So, the cross section is capital A and Young's modulus is capital E and u gives you the displacement field. Now, we propose a trial solution in the form of a polynomial and remember that it is a single solution that we are proposing which spans the entire domain. So, this trial solution should be valid for the entire domain spanning from x = 0 to x = L. Now, you can see these unknown coefficients k_0, k_1, k_2 and so on.

And you need to calculate those unknown coefficients based on the boundary conditions which are provided and also based on minimization of the residual. Now, you can show very easily that if you apply the boundary conditions which have been provided to you, the trial solution will simplify to a form like this k_2 into x square – 2Lx.

$$\hat{u}(x)=k_0+k_1x+k_2x^2$$
$$=k_2\left(x^2-2Lx\right)$$

It is because if you find out u(0) which is equal to 0 by the boundary condition, it is also equal to k_0 by the trial solution.

$$u(0) = 0 = k_0$$

Therefore $k_0 = 0$. Similarly, you can find out based on the gradient condition, so you have to find du/dx. That will be $k_1 + 2xk_2$ and you have to impose the condition that this is equal to 0 at x = L and thereby you would be able to evaluate k_1 in terms of k_2. So, k_1 will come out to be $-2xk_2$ and that needs to be substituted in place of k_1 in the trial solution and then you essentially reduce the trial solution to a single parameter trial solution.

Where the single parameter is k_2, which needs to be evaluated. Now, talking about the residual, what do you do in the residual? You try to find out that how much your solution deviates from the zero condition. That means, ideally in the exact equation, the left hand side equals to 0 while in the trial solution based equation where u is replaced by $\hat{u}$, you may not be exactly satisfying 0.

You have to make a very precise effort to go as close as possible to 0 and that is basically the idea of residual minimization. So, in this case, when you substitute the expression for the second derivative in terms of the trial solution, then you get the expression for the residual here and then you set the condition that that residual should go to 0, and then once you do that, you are able to find out what is that value of k_2 which will set the residual to 0.

Mind it that here you are able to satisfy the condition of residual going to 0 in the entire domain.

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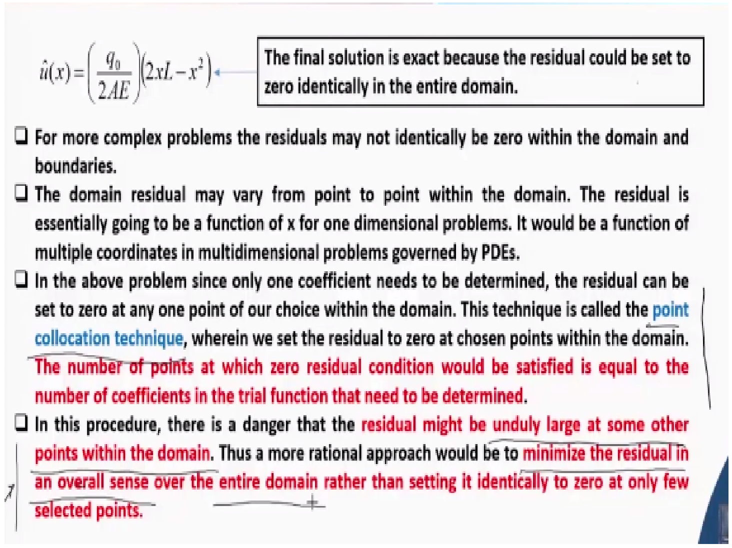

$$\hat{u}(x) = \left(\frac{q_0}{2AE}\right)\left(2xL - x^2\right)$$ ← The final solution is exact because the residual could be set to zero identically in the entire domain.

- For more complex problems the residuals may not identically be zero within the domain and boundaries.
- The domain residual may vary from point to point within the domain. The residual is essentially going to be a function of x for one dimensional problems. It would be a function of multiple coordinates in multidimensional problems governed by PDEs.
- In the above problem since only one coefficient needs to be determined, the residual can be set to zero at any one point of our choice within the domain. This technique is called the point collocation technique, wherein we set the residual to zero at chosen points within the domain. The number of points at which zero residual condition would be satisfied is equal to the number of coefficients in the trial function that need to be determined.
- In this procedure, there is a danger that the residual might be unduly large at some other points within the domain. Thus a more rational approach would be to minimize the residual in an overall sense over the entire domain rather than setting it identically to zero at only few selected points.

The final solution which you find is actually the exact solution to the differential equation. This was because you could set the residual exactly to 0 all over in the domain and therefore there was no error committed in obtaining the solution to the problem. Now, this was a situation where you could actually get an exact solution. For complex problems in finite element, you will not necessarily be able to set the residual to 0 so conveniently.

And therefore the residual minimization principle has to be always imposed and the attempt should be to best satisfy it as far as possible. Now, the kind of calculation we did over here falls in the category of a so called point collocation technique, where you actually try to set the residual to 0 based on a single parameter and in that case you may find that the number of points at which 0 residual condition is satisfied is essentially only one.

However, here we were fortunate enough that as we did that we could also set it to 0 all over the domain, but as we mentioned about more complex problems if we try to apply this principle, we may be able to set the residual to 0 only at a fixed number of points based on the number of unknowns that we have in the trial function, and then in the remaining part of the domain, we may not be able to satisfy the residual going to 0 exactly.

In fact, it may become unduly large at some other points in the domain, which is a big concern. Therefore, a much more rational way of pursuing the whole problem would be to minimize the residual in an overall sense over the entire domain rather than trying to set it identically to 0 only at a few select points. So, that gives rise to a more generalized procedure, which we call as the weighted residual technique. We will talk about the weighted residual technique shortly.

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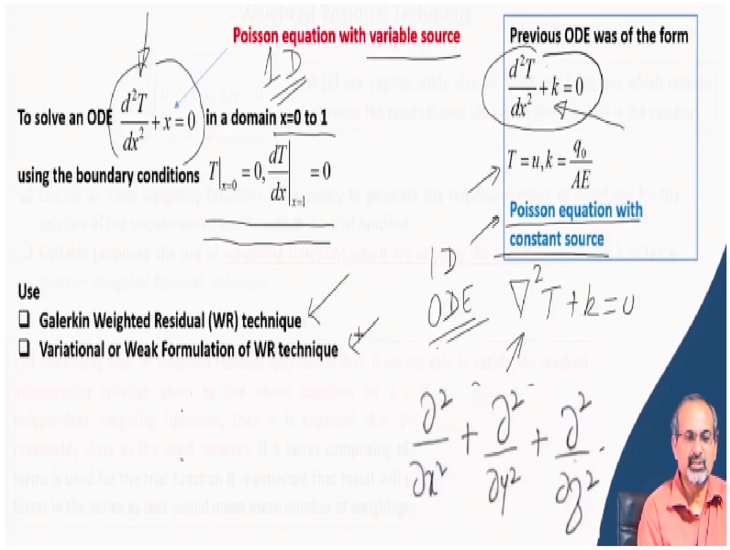

Before we do that, let us have a look at a problem which we would like to discuss now, which is more involved than the one we just solved. So, the one we just solved let us try to see what kind of ordinary differential equation it was. It had a second derivative in the dependent variable T and there was a source term k. In the problem that we solved if you were to just replace T with u and k by q_0/AE, you would be able to see the equation that you solved.

$$\frac{d^2T}{dx^2}+k=0 \qquad\qquad T=u, k=\frac{q_0}{AE}$$

So, it is essentially like a Poisson equation with constant source. However, here because you are doing it in one-dimensional space, it reduces to an ordinary differential equation, whereas as you remember that Poisson equation actually has an operator which looks more like this, where you have the Laplacian operator which comprises of partial derivatives like this. So, here because we are talking about one-dimensional problem, therefore it has essentially reduced to an ODE.

But it is analogues to the Poisson equation with constant source. Now, if you were to make that equation look a little more complicated, you could think about an equation of this kind,

$$\frac{d^2T}{dx^2}+x=0$$

where you are talking about a Poisson equation in one dimension with a variable source and then because we are handling one dimension, you see it in an ordinary differential form. We are taking a domain x = 0 to 1.

And we are imposing very similar looking boundary conditions like we did for the previous problem, but as we solve this problem, we would discuss first about the weighted residual approach which we talked about a few minutes back and then we will also talk about another approach which is called as the variational or weak formulation approach.

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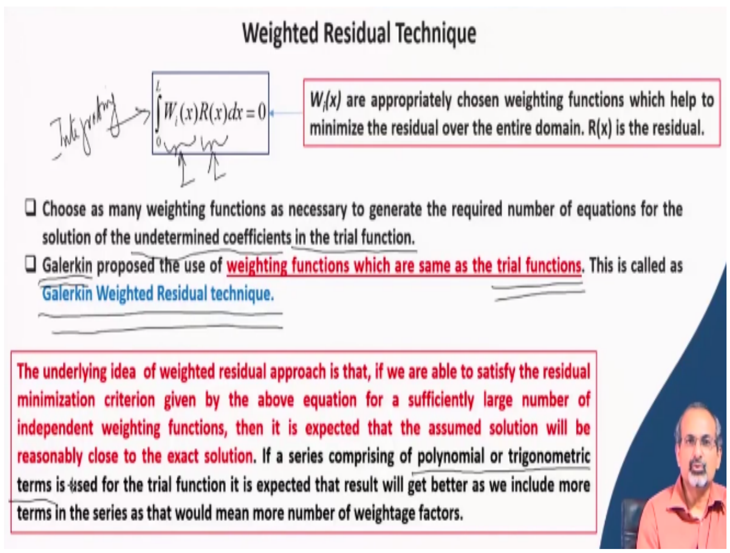

Let us look at the concept of the weighted residual technique. In the previous problem, we had talked about minimizing the residual in the entire domain. Here we are talking about minimizing the residual in the entire domain based on certain weights. So, we introduce weights. These are weight functions or weighting functions, which are multiplied with the residual function and they are integrated over the entire domain.

So, you are integrating a product of the weighting functions and the residual function and you are trying to set the integrated value to 0 in the entire domain, which would mean that all along the length of the domain you are trying to minimize the residual, not only at fixed points. Of course, you have to choose the weighting functions appropriately as you do that exercise.

$$\int_0^L W_i(x)R(x)dx = 0$$

You can choose as many weighting functions as necessary to generate the number of constraints that you would like to impose and that would exactly equal the number of undetermined coefficients in the trial function. Now, as you do that there is a very specific procedure which was proposed by famous scientist Galerkin. He proposed that the weighting functions which we use in this integral should come from the trial functions.

Therefore, we actually derive information about the weighting functions directly from the trial solution itself, which you are proposing. So, this is known as the Galerkin weighted residual technique. The underlying idea of the weighted residual approach is that we are trying to satisfy the residual minimization criterion for a sufficiently large number of independently set weighting functions and it is then accepted or rather expected that the assumed solution or the trial solution will be rather close to the exact solution.

So, more the number of constraints, more proximate will be the trial solution to the exact one. Now, if you try to imagine this exercise being done through the use of polynomials or trigonometric terms in the trial solution, then you may understand from intuition that if you include more number of terms in the polynomial or more number of terms in the trigonometric form using sines and cosines, you are likely to get a better and better solution.

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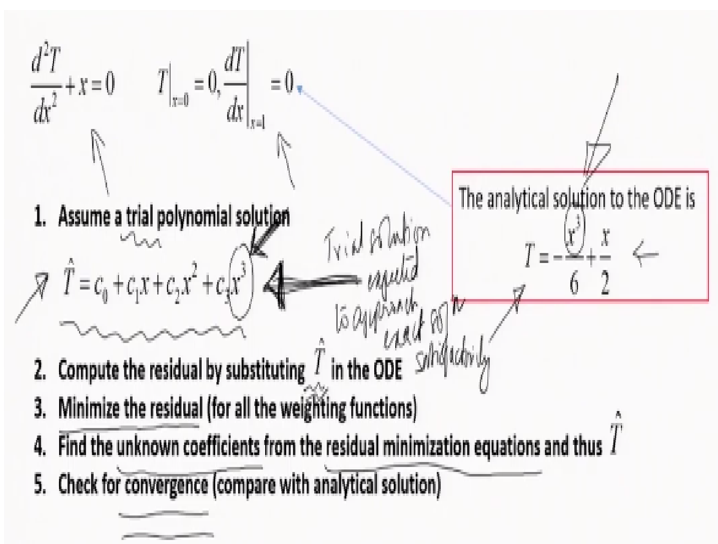

Let us look at the problem we are trying to solve here. We have stated the differential equation here with the boundary conditions here, and we will have a look at the analytical solution of the differential equation. So, if you integrate this equation twice and you impose the boundary conditions, it is a very simple exercise to show that this dependent variable T will vary in this form with x.

$$\frac{d^2T}{dx^2} + x = 0 \qquad T\big|_{x=0} = 0, \frac{dT}{dx}\bigg|_{x=1} = 0$$

$$T = -\frac{x^3}{6} + \frac{x}{2}$$

So, that is based on the two boundary conditions that we have imposed. You have a trial solution over here which looks similar in order in terms of the highest order polynomial over here, we can expect that this solution may approach the exact solution fairly well.

$$\hat{T} = c_0 + c_1 x + c_2 x^2 + c_3 x^3$$

Of course, here we have the convenience of having the analytical solution of the ODE available to us.

Based on which we are able to guess that it would be rational to include this highest order term x^3 as is seen in the exact solution to be included in the trial solution. We may not always be very fortunate to have the analytical solution available with us readily. In that case, it is rather difficult to guess that what should be the minimum order of the polynomial.

So, based on experience, we have to choose the minimum order of the polynomial in such situations. Having said that, we begin the whole exercise by judiciously assuming the trial solution in the form of a polynomial and then we compute the residual by substituting the trial solution in the ODE like we did in the previous case. We then minimize the residual for all the weighting functions that we have in the trial solution.

And then we try to find the unknown coefficients from the residual minimization equations. Then, we check for convergence in the sense that how good is the approximate solution in terms of the analytical solution, which we already have available with us.

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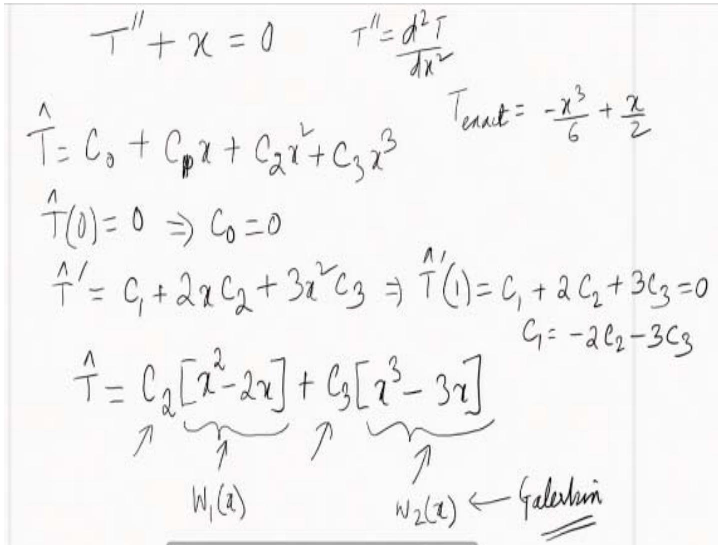

Let us do this exercise a little bit more in detail. So, we have this equation $T'' + x = 0$ where T'' stands for d^2T/dx^2 and we also remember about the boundary conditions that we have to satisfy. Also remember that the exact expression

$$T = -\frac{x^3}{6} + \frac{x}{2}$$

because at the end of our Galerkin based weighted residual method, we need to compare how good the solution was with respect to the exact one.

So, based on the trial function that we have assumed, let us try to work out the steps. We are now imposing the boundary conditions. We are finding the first derivative and then we are imposing the condition that at x = 1, the value of this derivative will be 0.

$$\hat{T}'' = C_1 + 2xC_2 + 3x^2C_3 \Rightarrow \hat{T}'(1) = C_1 + 2C_2 + 3C_3 = 0$$
$$C_1 = -2C_2 - 3C_3$$

Based on that we get an expression for C_1, we substitute that expression for C_1 in the $\hat{T}$ expression. So, you can understand that already C_0 is 0 and you are now replacing C_1 in terms of C_2 and C_3.

So you will finally have an expression for $\hat{T}$ in terms of two unknown coefficients C_2 and C_3 and their coefficients or rather the functions which are accompanying them will be the weight functions or weighting functions. We will call them as W_1 and W_2 respectively. Of course, both are functions of x. Remember that this consideration is based on what was proposed by Galerkin that the weighting functions would come from the trial functions.

Therefore, we do not have any separate proposal for the weighting functions; we are deriving it directly from the trial functions.

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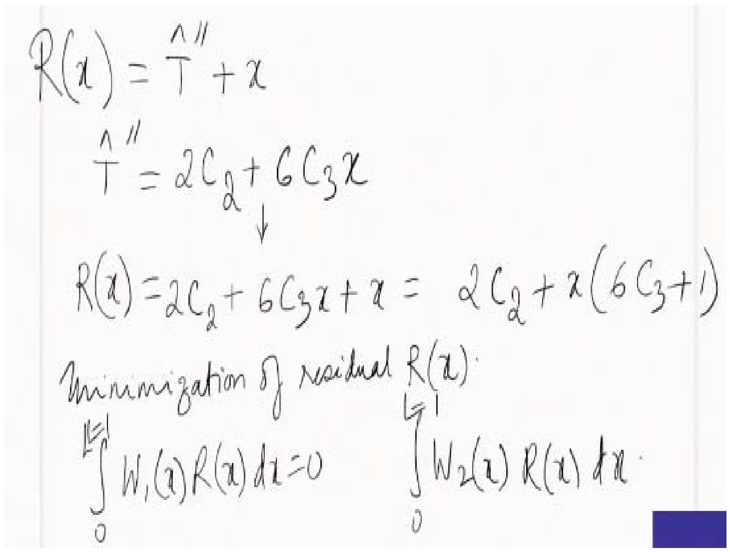

Let us go ahead with the next step. We will say that the residual is equal to $\hat{T}'' + x$.

$$R(x) = \hat{T}'' + x$$

So, what is $\hat{T}''$? You have to take the derivative of the $\hat{T}$ equation twice and then you will have this expression for $\hat{T}''$.

$$\hat{T}'' = 2C_2 + 6C_3x$$

You are now ready to obtain the residual function. So, you just substitute this expression and add x to it. So, now you have an expression for the residual.

$$R(x) = 2C_2 + 6C_3x + x = 2C_2 + x(6C_3 + 1)$$

Once you have an expression for the residual, you are ready for the minimization problem and we have two weighting functions and therefore, you would write two equations in the form of integrals for minimizing the residual. Let us see what they are. So one would be based on W_1, the other would be based on W_2. In fact, here L = 1, so you can as well write 0 to 1.

$$\int_0^{L=1} W_1(x)R(x)\,dx = 0 \qquad\qquad \int_0^{L=1} W_2(x)R(x)\,dx$$

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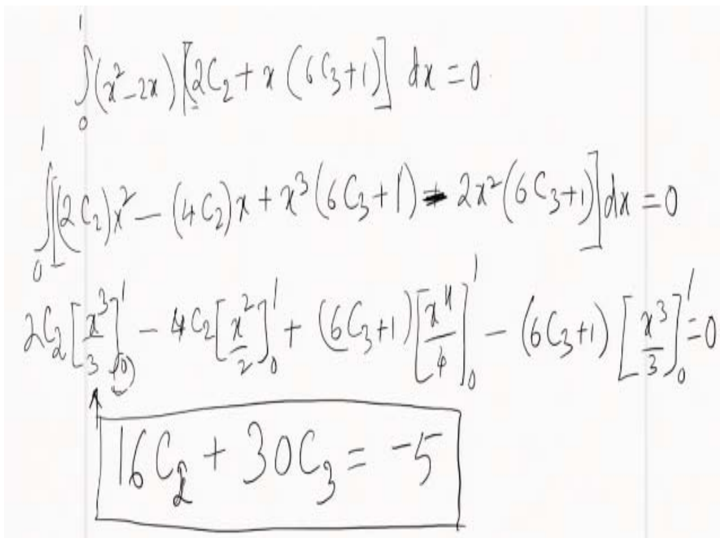

Now, let us take up the solution of each one of these integral equations one by one, so that we come up with relevant equations involving C_2 and C_3 and we solve them simultaneously after that. So, let us look at the first integral. So the first integral will look like,

$$\int_0^1 \left(x^2 - 2x\right)\left[2C_2 + x\left(6C_3 + 1\right)\right]dx = 0$$

Now we integrate term by term.

So these are constants, so they will go out of the integral, and these are definite integrals, so you have to put the limits. As you can see, each one of these after you put the limits will yield one one term because the lower limit is 0, so there will be no contributions. So, once you do that and you finally rearrange the terms, you should be able to come up with this equation. This is the final equation we get out of this exercise. So, this is the outcome of the first residual minimization equation.

$$6C_2 + 30C_3 = -5$$

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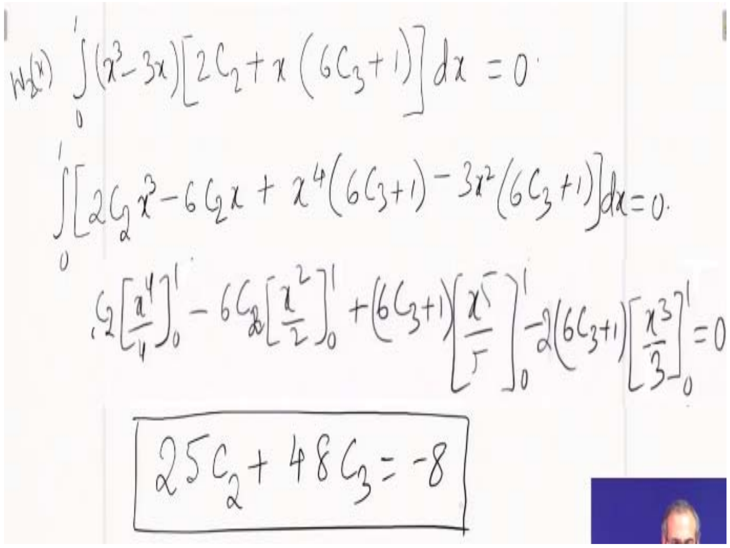

Let us go to the next residual minimization, which is based on W_2. So, let us write down the problem.

$$\int_0^1 \left(x^3 - 3x\right)\left[2C_2 + x\left(6C_3 + 1\right)\right]dx = 0$$

In a similar manner like we did in the previous case, we expand this and we integrate. So, you can actually take 3 out here and then finally you will get this equation as your outcome after the integration is completed.

$$25C_2 + 48C_3 = -8$$

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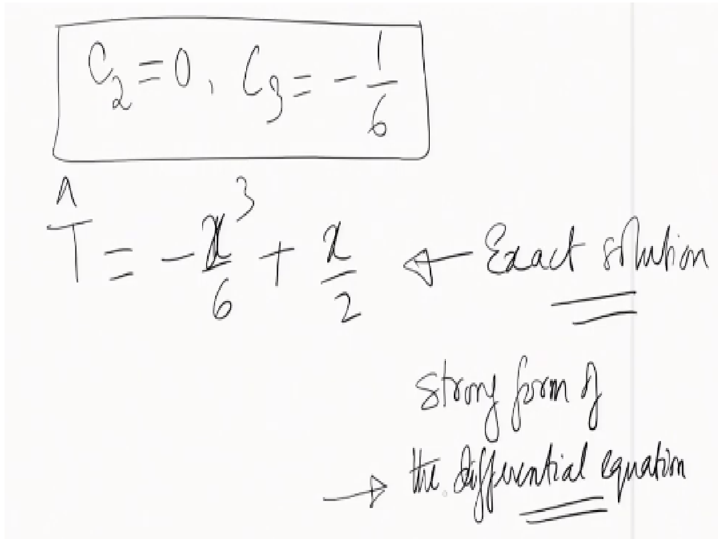

So, what do you have as an outcome of these weighted residual minimization exercise? You have 2 linear algebraic equations in 2 unknowns C_2 and C_3. So, if you solve for these 2 unknowns, the values that you can get are $C_2 = 0$ and $C_3 = -1/6$. So, this is what is going to come through simultaneous solution of those 2 linear algebraic equations, which means that the $\hat{T}$ comes out to be this, which is the same as the exact solution.

$$\hat{T} = -\frac{x^3}{6} + \frac{x}{2}$$

Which means the weighted residual technique that we just discussed about based on Galerkin formulation works fairly well. Remember that, here we have used the so called strong form of the differential equations. That means, we have dealt with the differential equation with the highest order derivative that it had and worked out the trial solution based on that highest order derivative.

We will discuss about another technique which is called as the variational formulation or the weak form of the weighted residual technique in the next lecture. Thank you.