Introduction of Aerodynamics

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Module No. # 01

Lecture No. # 05

Forces in Fluids

So, before proceeding further, I would like to mention that, there are some seriously, it seems, I am not very sure. It is in application of that transformation from surface integral to volume integral. So, I have written something wrong and you have not pointed out may be because of non-familiarity with the notation. So, first I will correct that.

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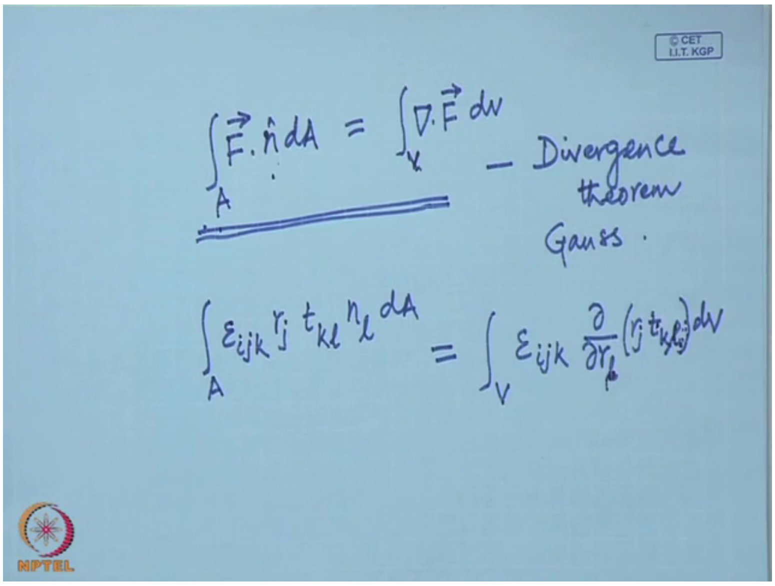

You know that conversion from surface integral to volume integral is, what is commonly called as the divergence theorem, is given by integration of any vector. Let us say F over the surface of any volume, is divergence of that vector integrated over that volume. The relationship between this volume and area is, this area is the surface area of this volume. They are not just any arbitrary area and volume. This volume is enclosed by this surface or this surface is the surface of this particular volume. So, this is what is the relation and commonly called as divergence theorem or gauss divergence theorem. This is what is called divergence theorem, also called gauss theorem.

Now, this is the rule that we applied while evaluating the moment due to the surface forces. Now, the surface forces were, if you remember, if you look back to your yesterday's note or last class notes, that the surface force, please correct me, if I am using, if I am not using the notations as it was in the last class. Epsilon i j k r j t k l n l d a integrated over the surface. This part is all right. This, we are applying this divergence theorem, which resulted or which will result epsilon i j k, divergence r j. It seems that perhaps, while writing this, I wrote n l also on this right hand side. If I wrote n l here, then remove it. Because, that n l will not be there on this. This comes directly from this, this relation. And because, of this n l the divergence here is with respect to this r l. The divergence is written as r l. You can which is the second subscript for t. So, that is the only correction, that this n l will not be there. The n represents the normal to the surface. So, if the right hand side is not surface integral, it is a volume integral. So, there is no question of that n. And that n is being taken care of by this r l. You can see that, this has the same subscript what n had, which is the again the second subscript of this t k l. So, that small bit of correction. Anyway the result, what we found, that of course, it is not changing because of this. I think in this statement, it was a slip that I wrote that n l but, the subsequent steps, of course, I did not. Anyway, and the result that we found that the stress tensor is symmetric.

Now, I hope or I think, it is correct that you have already come across, if not stress tensor in direct details but, the moment of inertia tensor; that is also another tensor which you have come across earlier. This is a general property of any tensor that if you change your axis system, particularly if you rotate the axis system, then the element of these tensor will change. Like this stress tensor t k l, it has the component t 1 1 t 1 2 t 1 3 and so on. These components will change, if you change your axis system, particularly if you rotate your axis system. And there is a particular axis axis system. With respect to that, all the off diagonal terms, that is t 1 2 t 1 3 and so on, all of them become zero. Only, the diagonal elements remain. One particular axis system, if you go on rotating say, you have a particular axis system, based on that axis system, you have evaluated everything. Think about the moment of inertia which is simply evaluated as something square of the distance into d m. And of course, the square of, now if you rotate your axis system, these elements will change. At one particular orientation of the axis system, you will find that all off diagonal elements are become zero, only the diagonal elements remain. This particular axis system is called principle axis system. This also, I think you have done your earlier mechanics course. In connection with stress, principle stresses, principle strengths you have found. Now, tensors also have a very important property, which are known as invariance. A second order tensor has three invariance. A second order tensor means which are mathematically represented by a three by three matrix has three invariance. Invariance means which do not vary as we rotate our axis system. They are called invariance. And a second order tensor, that is three by three tensor has three invariance. So, they are called the tensor invariance.

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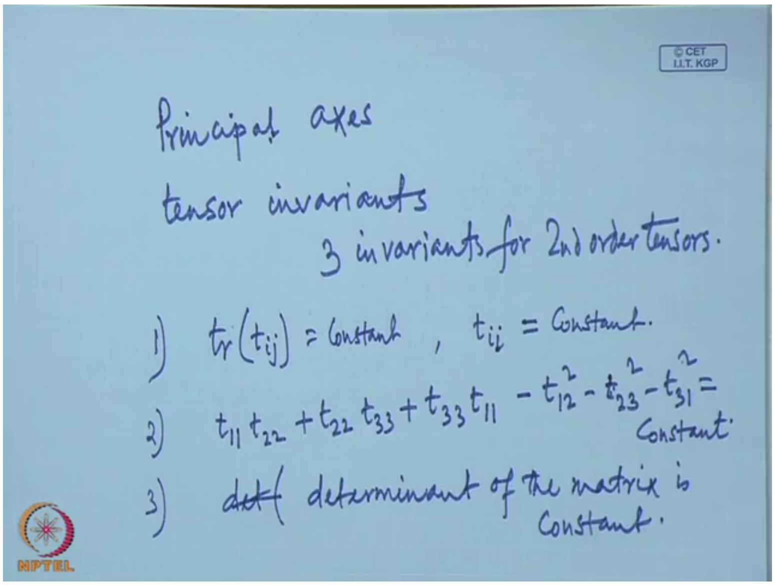

So, we have the tensors, have principal axis system. These are certain properties of all tensors. Then, tensor invariance tensor invariance, a second order tensor has three tensor invariance, three invariance for second-order tensor. Now, see the tensors are represented by matrix. The tensors are usually represented by matrix but, you should remember the there is some difference between a matrix and tensor. Since, tensors are represented as matrix, so, any type matrix operation that you can perform on matrix, that can be performed on the tensors also. The main or the basic differences comes from here. The tensor is matrix is just a collection of numbers. Those numbers do not have any meaning. When we call it is a matrix, those numbers do not have any meaning. They are just collection of numbers; however, the tensor is a physical quantity, tensor is a physical quantity. So, its elements has a very definite meaning, like in this case of stress tensor t i j. as you say, mention, that t i j represent the force per unit area along i direction, acting on a surface which has j its normal. But, if you just write a matrix, then the numbers whatever there in the matrix, the elements of the matrix, they do not have any other meaning, they are just number.

Now, the tensor invariance, of course, the we will be only using the first invariant, not the second and third invariant. But, since we have mentioned, we will tell you, what are the three invariance. The first invariance is what is called the stress; that is sum of the diagonals. It is invariant; that means, as I said that, if you rotate your axis system, the tensor elements will change. There is a particular orientation, where no off diagonal elements are there. All off diagonal elements are zero. Only, the diagonal elements is there. But, whatever the change is, the change will be always such that the sum of the diagonal elements will always remain constant; that is, let us say, if you have stress tensor each component is representing force per unit area along a particular direction acting on a particular plane. Those individual component will change, if we change on keep on changing our axis system, particularly, if we keep on rotating; however, whatever those individual elements are, the sum of the diagonal will always remain fixed constant. So, that is the first tensor invariant. The stress of the tensor or stress of the matrix representing the particular tensor is always constant. That is, first is the trace usually denoted by t r. Trace of t i j is constant. That is, the trace of t i j is what, t i i. when I write t i i, this index i is repeating. So, it means t 1 1 plus t 2 2 plus t 2 3 t 3 3. This is constant. t 1 1, t 2 2, t 3 3; all three of them can change but, we will change in such a way that the sum will remain constant.

The second tensor invariant is that sum of the principal minus is constant which in this case can be written as say, the second invariant. Let us denote this diagonal elements only or let us think about this t matrix. t 1 1 into t 2 2 plus t 2 2 into t 3 3 plus t 3 3 into t 1 1 minus t 1 2 square minus t 2 3 square minus t 3 1 square is constant. This is true for any tensor, not only that stress tensor. Any other tensor, like say moment of inertia tensor or strain tensor or any other tensor, you which you may come across, it is true for all of them. Whenever is a second-order tensor, these the there are three invariants and these are and the third is determinant of the matrix formed by t i j. How do you write? Say, t we have already used for force, so, determinant of let us simply call it determinant, instead of writing by any notation because, we are not going to use it. Determinant of the matrix is also constant. This is the third invariant. So, if you have any tensor, these are the three invariants of the tensor. So, when we rotate the axis system, so that the elements of these matrix or the individual stress elements are changing but the change is such that the sum of the diagonal always remain constant. Now, what are these diagonal elements? If we look to this matrix or let us write in a matrix form, the stress tensor in the matrix forms.

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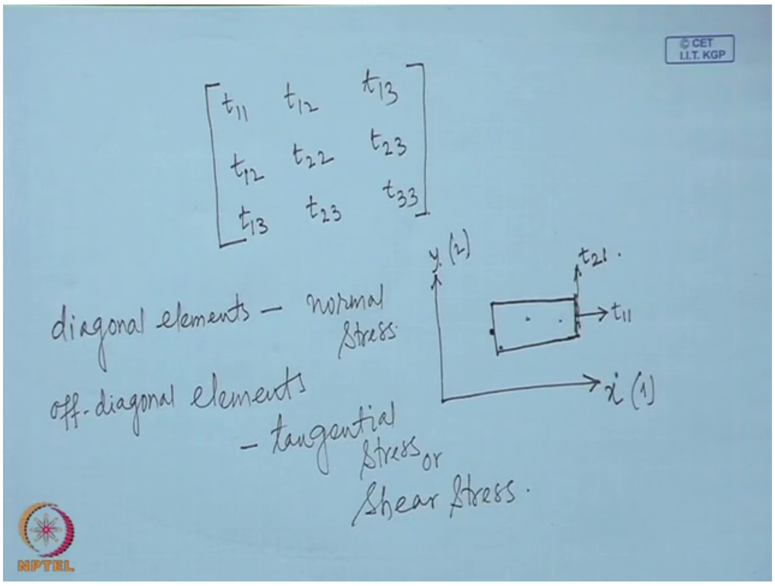

The stress tensor, if we write in matrix form is t 1 1 t 1 2 t 1 3 t 2 1 which is same as t 1 2. So, we are not going to write its t 2 1 but, we will write as t 1 2 t 2 2 t 2 3. t 3 1 is same as again t 1 3 because of symmetry. So, we will again write t 1 3. t 3 2 is again same as 2 3 and t 3 3. Now, t 1 1 is the force per unit area acting along direction one, on a surface to which direction one is normal. Now, think about say, in terms of x y z because, that might be little more comfortable to you. That is, if we think it is t x x, then it is x component of the force per unit area, of course. Acting on a phase, which is normal to x. So, what is this force? This is an force normal to that plane. So, these all these diagonal components, they are they represent normal stresses, normal stresses. The stress which is normal to the phase, on which it is acting. The others are t 1 2, we can say very easy, it might be easier to comprehend using a two-dimensional element. In a two dimension, an element is simply, so, a simple rectangular, make a smaller rectangle, instead of making such a big rectangle. Let us say, for your convenience, we say the x y. So, the t x x will be in this direction acting on this phase, this because x is normal to this phase, also to this phase. The direction, it might be along this or along this. We are not worried about the direction at this stage, whether it is towards the phase or away from the phase. But, this is normal to the phase. What about t 1 2? You can see that, that will be something like this. So, if we call this t 1 1, this x is 1, y is 2. Now, so, these off diagonal elements are all tangential stress, also commonly called as shear stress and the diagonal terms are the normal stress. And we know that, normal stresses are either tension or compression. The normal stresses are either tension or compression. And the off diagonal elements are also at tangential stress or shear stress. So, diagonal elements- they represent normal stress, off diagonal elements- tangential stress or shear stress.

Now, with this, we will see, what is the meaning of the definition that, we initially proposed for our fluids. We defined fluid are substances which cannot withstand any tendency by applied forces to deform it, in such a way that leaves the volume unchanged. Or other way that, if we apply a force,then fluids are those substance which would not be able to the tendency of this force to deform it without changing the volume; that means, it can withstand but, in that process it has to change its volume. Without changing the volume, it cannot deform. No such deformation is possible which will leave its volume unchanged. When a force is applied, it will deform, all right. But, the deformation will be such that the volume will not remain unchanged. The volume has to change. That is the definition, we proposed for fluid. And we will see, what is the meaning of that, in terms of this stress for that. First of all, we will consider this fluid is at rest.

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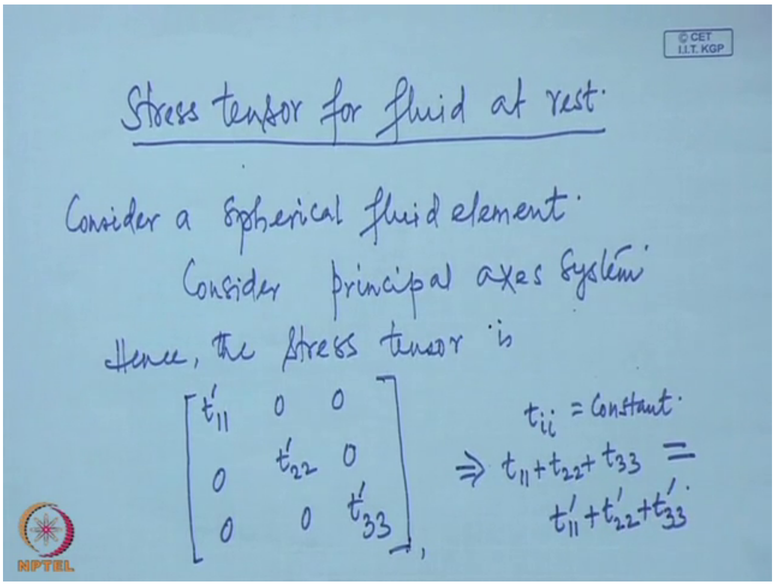

Let us consider fluid at rest and for simplicity. So, we will call stress tensor, we will would like to see, how the stress tensor is for fluid at rest. For convenience, let us consider a spherical element. Now, consider a spherical element. Sufficiently small, this fluid element is sufficiently small so that, we can consider the stress tensor is more or less uniform. We can assume. We will later on see what it is. But, we will assume that a small fluid element spherical element so that the stress tensor remain more or less uniform and we will choose the principal axis system for stress. That we can choose without losing any generality. That we are choosing, only the principal set of axis, set of principal axis system. So that, the stress tensor contain only diagonal element. So, consider again principal axis system. So, the stress tensor is what we have already, you know, that we have written all t 1 1, 0, 0, 0, t 2 2, 0, 0, 0, t 3 3 and to make that, this is principal system and different from the other, let us denote a prime. That, these are now the principal stresses; however, we know that, this is an invariant t i i equal to constant, once again t i i means t 1 1 plus t 2 2 plus t 3 3 and that is constant; means the same is t 1 1 prime plus t 2 prime plus t 3 3 prime. So this implies, that for any condition t 1 1 plus t 2 2 plus t 3 3 equal to t 1 1 prime plus t 2 2 prime plus t 3 3 prime.

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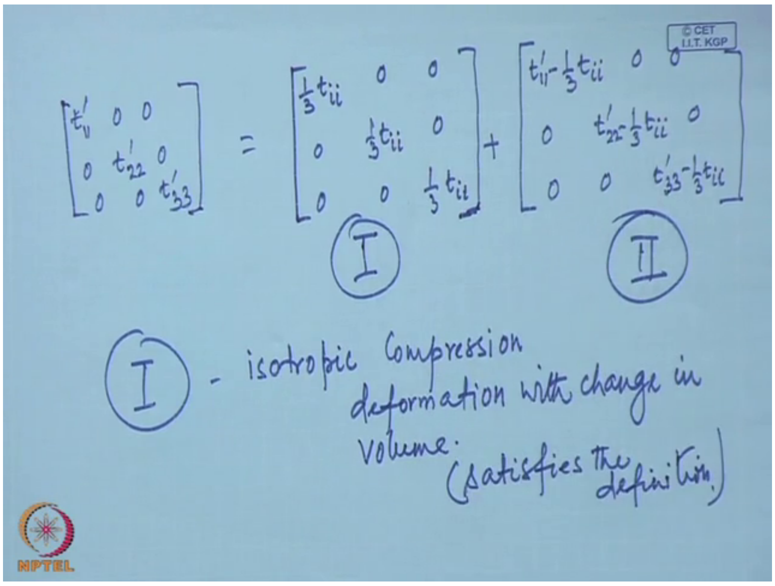

Now, we would like to write this stress as sum of two stress tensor, sum of two stress tensor; that is, we would like to write, let us write it explicitly instead of using any notation.

This is simple. So, if we say that this is the stress that is acting on the fluid, we can say that, that mean, these two stresses are acting on it. In which, in one case, you have made it one-third of the sum and the whatever is left out, that you have to consider as a second stress system. Now, look to this first stress system. What it is? This is equal in all direction. This is a this has a spherical symmetry or isotropic behavior, equal in all direction. And since, all are same, this means that all three are either tension or compression. As it happens to them, the actually compression. That is what, fluid can take at rest. It cannot take tension, anyway. So, but, all three are same. Our experience says that this is compression, that all three are compression. Now, what happen, if that type of stress is applied on a fluid element, a spherical element? In which, an isotropic compression is applied, equal amount of compression in all direction. Its volume will change. In case of compression, it will be compressed, volume will decrease. So, this deformation, resulting in, change in volume. So, according to our definition, this is permitted; that is, an isotropic state of compression is allowed or definition of fluid permits it.

Now, look to the second one. We will call it say, this denote it say, one and two. So, one represent isotropic, we are calling it compression from our. So, this is deformation with change in volume, which is permissible by, so, satisfies the definition, satisfies the definition. Definition of fluid that, we have proposed. Now, look to the second one. The sum of this diagonal or the stress of this matrix is zero. If you sum the diagonal, it becomes zero. Then, what does it mean? All three are normal stresses and sum of the normal stresses are zero; obviously, it implies that at least one at least one is tension and at least one is compression. The third one may be either tension or compression. All three cannot be same. At least one is tension, at least one is compression and the third one might be either of the two, so that the sum becomes zero.

Now, if this type of force is applied on a spherical element, what will happen? In one axis, along one axis, you are applying a tension. Along another axis, you are applying a compression and along the third axis, you are applying a tension or compression. Such that, the sum total of this tension and compression is zero. Yes, volume will not change because, the sum, total of the compression and tension is zero. So, the total volume will remain unchanged but, as far as deformation? Deformation will occur. What type of shape, the sphere will become?

Ellipsoid, sphere will not become cylindrical. It will become ellipsoid. You have stresses from all these directions. So, this second part imposes that the deformation will be such that, there will be no change in volume. And our definition of fluid does not permit that; meaning that the second part of the stress, will not be possible. That will exit in a fluid at rest.

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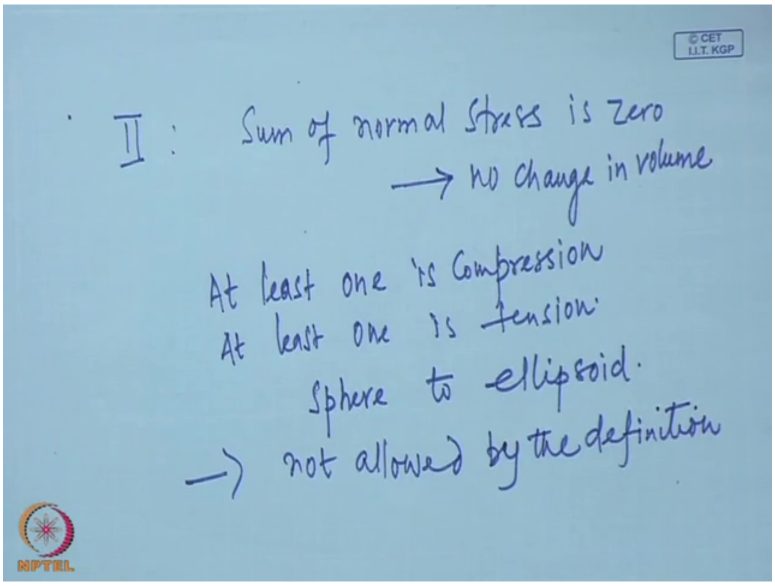

So, write the second part, sum of diagonal, so, sum of normal stress, the result, sum of normal stress is zero. No change in volume. At least, one is compression. At least, one is tension. The third is either of the two. So, deformation is sphere to ellipsoid. Deformation without change in volume, so, not allowed by the definition that we have. So, what is the meaning of this, then? That when a fluid is at rest, the stress system has to be isotropic, equal in all direction and as it happens, this is compression. And it is all axes system or principle axes system. All axes system are principle axes system. For a fluid, at rest, all axis system are principle axes of stress. And at any point at any point the stress is only isotropic compression; that is, equal form all side. A compression equal from all side, from all direction and all directions are principle direction. As it happens, this isotropic stress is compression. A fluid at rest cannot take tension.

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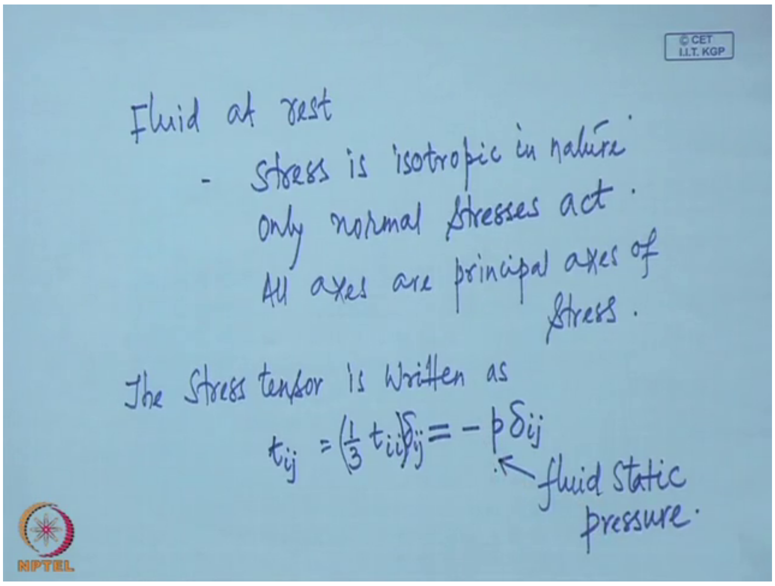

For fluid at rest, stress is sorry isotropic in nature. Only, normal stresses act. All axes are principle axis of stress and we can write the stress system, the stress tensor. And in the second term, delta i j is written to make it a mathematically compatible equation because, one-third t i i is just a number. This t i j represents a tensor but, one-third t i i is just a number. One-third of t 1 1 plus t 2 2 plus t 3 3. So, this delta i j is given. So, that it has the tensor form. And this one-third t i i, since it is equal in all direction, we do not need write all the time, t i i and all. We can just write one number, which we have written here p and call this as fluid static pressure. Perhaps, the word hydrostatic pressure is more familiar to you because, say water was perhaps the most common fluid or studied most, the fluid that has been studied most. And pressure and all these things were usually encountered with water. So, since long this pressure is referred to usually as hydrostatic pressure, we are making it more general fluid static pressure. So, this is the definition then, of static fluid pressure. That, it is one-third of the sum of the normal stresses, acting on a fluid at rest.

I think you have come across another definition of pressure in your thermodynamics. I do not know, whether you had a formal course on thermodynamics or not but, you might have the course on kinetic theory in school physics. And in that context, you defined something called pressure. Coming because of the collision and momentum transport, may not be in that details. But, because of the collisions the so called properties of the substance or in that case perhaps, you studied only gas, which is also fluid. So, properties of fluid, particularly, the pressure and temperature, all are basically manifestation of those molecular motions and their collision. So, that is a thermodynamic pressure. This, we have got something, based on a stress consideration or let us say dynamic or mechanics consideration. So, that is a thermodynamic pressure, this is a mechanical pressure. Is there any relationship between the two? We will not answer that question now. We will leave it for later. But, we will later on come at what is the difference between this pressure and that pressure or is there at all any difference or both are same? but, this of course, we will discuss little later. Now, the definition of that fluid, that we initially postulated has become clear that, what we intended as a definition of the fluid is that the stress system for fluid at rest must be only an isotropic compression. And that is what, is the implication of that definition. That fluid, while at rest takes only an isotropic compression.

Now, let us move on to the equilibrium. We have already discussed that two forces are acting in general on a fluid at bulk. If we consider certain bulk amount of fluid, there are two forces acting on it and when the fluid is at rest, of course, these two forces must balance each other; the body force and the surface force. So, say for fluid at rest in equilibrium or equilibrium consideration of at rest is has to be in equilibrium. So, that is actually super flow (( )) fluid at rest in equilibrium, at rest has to be in equilibrium. So, we are considering now, equilibrium condition. Now, the forces acting, the body force. How much is the total body force? Consider a fluid of volume v, which is bounded by a surface, then what is a total body force acting on it?

Total body force is, we have already defined the body force to be rho f d v. So, when this is integrated over the entire volume, this gives the total body force. Let us call this to be capital f and it is, if you want, you can use even this vector notation. Otherwise, you can forget. And the total surface force acting on the same fluid, total surface force acting on the same fluid, how much it is? Yes, now it should be straightforward to write that. Earlier, we have written it already, tau i j n j d a, but now tau i j n j has a special meaning. The tau i j, we have got a definite value. Yes, the tau i j has become minus p. The tau i j has now become minus p. So, you can simply write that p. Integrated over the surface which encloses this volume v. And again, we can apply this divergence theorem because, these are these you need to equate, you need to balance. So, they should be of same type of quantity. So, this becomes, in terms of volume integral. What it is? And since the fluid is at rest, the sum of these two forces is zero.

Remember, there is something about the volume, this v. This surface or this volume, since we are considering in the first relation itself, the total body force is v integrated over this volume; that means, this volume must be entirely within the fluid. You cannot consider a volume which is partly in fluid and partly not. The volume must be such that, it is entirely within the fluid. And since, this relation is general for any arbitrary volume; we can say that the integrand itself is zero. So, that is or to give our notation, this f is a vector. So, we will write in vector notation rho f i, this is plus. And as you know that the body force is often the gravitational force, that is, this f is g and you can have your axis system aligned in such a way, that the g has only one component. In that case, see, this is actually representing three small differential equation, for each i equal to 1, 2, 3. It represents three small differential equation d p d x 1 equal to rho f 1 and so on. And you can chose your axis system, particularly, when the body force is simply gravitational force and you know it has a, only in one direction. So, you can make align your axis along with that, So that only one equation will have nonzero right hand side. The other two equations will have zero right hand side.

Now, for those cases where the body force, f is a potential function or a conservative force field, let us say for conservative force field, let us write this or your where size what? The potential energy per unit mass, the potential function, is it not? The potential energy associated potential energy per unit mass, associated with this conservative field. And if we put it here, then what do we get? For conservative force field, the equation then become, write this gradient notation. If we take a curl of this, you are familiar with operator curl. So, if we take a curl of it, what will be the result? If we take the curl of it,what will be the result? Curl of a gradient, curl of a gradient is always zero. Curl of a gradient is always zero. The curl usually, written in this fashion. The minus of course, we can take it and this further becomes…

Now, see starting from this equation, if you start from this equation, this equation may indicates that there will be certain surfaces on which the pressure will remain same. If you consider a fluid under different condition, then there will be certain surfaces on which the pressure will remain constant and so will be density. These surfaces, we may call it, level surfaces because, the pressure is same on that. So, there are level surfaces for pressure, similarly, level surfaces for density and also level surfaces for psi. Psi, as we mentioned that, this psi function represents potential energy per unit mass associated with this, particularly, this force field. Again, you know that on there are surfaces on which the potential energy is fixed. Like think, if you think in terms of the gravitational potential which is measured from the center of the earth. So, at any fixed altitude, the potential energy will remain same, which is not exactly or horizontal surface, if you think, in terms of the r, it is a spherical surface. So, similarly, there are level surface for pressure, there are level surface of density, and there are level surface for potential energy. And looking to the last equation, this curl gradient of rho, cross gradient of psi equal to zero, what does it mean? Gradient, the result of gradient operator is always a vector. So, gradient of rho is a vector, gradient of psi is a vector. So, this two cross product is, cross product of two vector is zero, everywhere.

So, that means, this says that level surface of density and level surface of this potential energy, they will coincide. The last relation have the meaning that the level surface of density and level surface of this gravitational or the potential energy will coincide. Also, the first relation or the first relation on this page, this, this gives one more information; that the level surface of pressure is normal to the direction of the body force. Level surface of pressure is normal to the body force. That comes from this equation a level surface of pressure is normal to the body force.