Introduction to Helicopter Aerodynamics and Dynamics

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Lecture No. # 25

So, what we have is I think three flaps, two lag, one torsion, one axial. About r 3 appear 3 2 1 1 3 2 1 1 1 or 6 or 7, 8 modes, or 9 modes. We take r 4, flap 2, 3 lag torsion then we do system.

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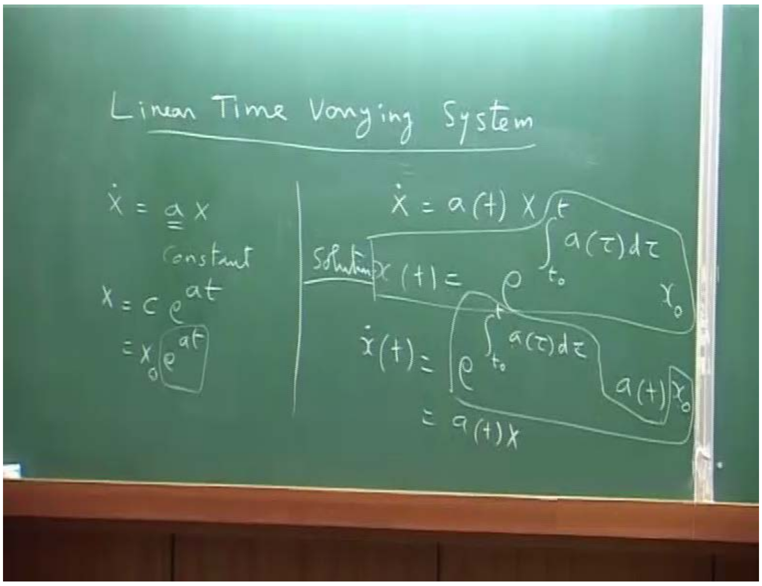

See we all learnt very simple equation; first order equation if you take x dot equals a x a is a constant this you write immediately the solution as x is equal to some constant e to the power what right x of t. This is the solution; the constant is the initial condition may be at equal to 0 x is equal x naught. So, you may write this as basically x naught e to the power a t.

That means this particular term as time evolves from 0, is changing the initial condition in the sense not that change initial condition is changed; initial condition is getting multiplied to sometime varying function gives you the response that any specific time. Similarly, let us now go to this equation. Now, how will I write my solution a f t can be it is still a linear system it can be any time varying function now if you write assume this solution x of t as e to the power integral t 0 to t a tau d tau and then x 0 x 0 is a initial condition.

This is the solution to this equation because, if you differentiate this what will happen? x dot e to the power. So, that will stay as it is. So, e to the power minus t 0 t a tau d tau then you differentiate this, but that will be nothing, but a t and x naught and since this term this is x t. So, you will find a of t x. So, you stay this is my solution all right now you just say similar I said this is x naught if the initial condition and there is a time varying function which is getting multiplied to initial condition to give you response at any particular time.

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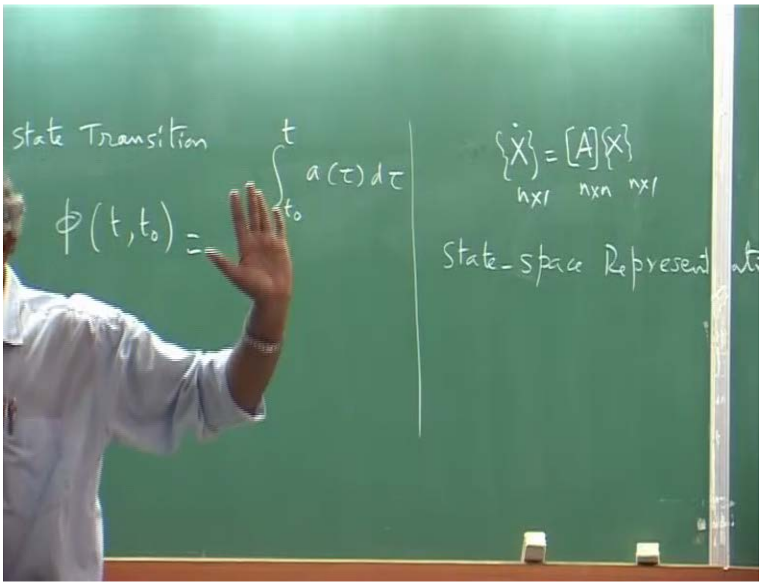

This is called state transition; if it is a matrix, it is a matrix problem. Here, it is just only function and this state transition phi is e to the power integral some t 0 to t a of tau d tau. Now, one key difference between this and that here, this is also you can write if a is a constant. If a is a constant, what will happen? This will come out; you will have t minus t naught. But, t minus t naught is one term; that means, the difference between the initial and the response at any time.

So, if it is a constant thing you will find the response is only the difference between the times. Whereas, if a is a general time function you will find t naught as well as t, but they it will not be the difference; it need not be the difference. It is a function of both t and t naught; it is not the difference between the initial and final time that is determining the response. This is the character and this is called the state transition in the matrix form because we are, we do not solve this problem matrix you called a state transition matrix.

First order one variable you can solve. Now, you extend it to matrix equation; that is the starting point. Why we have to go for matrix formulation is, I am writing now you take it x dot is a x. This you all know if a is a this is a, please note this can be n by 1. This you must have studied; you do not actually, this is called the state space representation or a first order representation. This is basically called state space because, if you are given a ODE linear all we are talking linear; if you are given a ordinary differential equation of any order please understand, it can be second order it can be third order can be nth order you can convert that into this format always I will just do that little bit.

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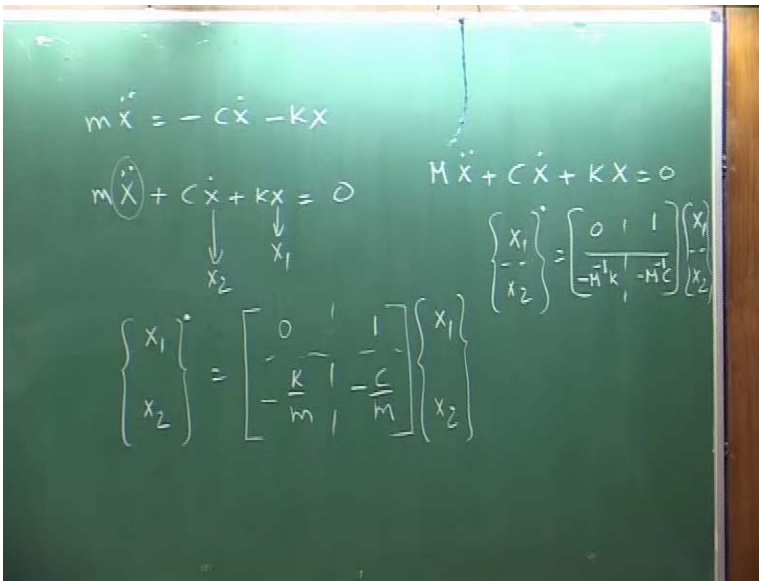

So that you know that, suppose you take the simple equation you are all familiar with the second differential equation m x double dot a x is a let us take it 0 by f of t means otherwise that will be a forcing function. Now, this is a second order equation I can have third order also. How will I write in this form? This is what the state-space representation is.

What you do is, there is a very clear simple thing; you start, you call this as x 1 you call this as x 2. Now, your equation x 1 x 2 dot time derivative that is x 1 dot x 2 dot x 1 dot is what X 1 dot is basically x 1 is x, x dot is x 2. So, x 1 dot is x 2 now x 2 dot is nothing, but x double dot because x 2 dot, but x 2 is x dot therefore, that is nothing, but x double dot. So, you can this stage you take this equation minus e x dot minus k x right; you know this is x 1 this is x 2 this is x 2 dot. So, you will have minus this is x 2 this is k by m minus c over m that is all.

This is the these takes place form because what we have done is why state is in this case we are talking about possession is a state velocity is another state. If I know possession and velocity then, I know the other derivative because if I know x and x dot I know x double dot is not that I have to solve separately for that. Do you understand? But, I must know x and x dot, is it clear? Now, you see this can be converted into matrix form also what you do is instead of single values you may have m x double dot. These are all matrix; now how will you get the form is you will put like this x 1 x 2 because, this is again the vector. So, you put that you will put 0 1 and here you put minus m inverse k minus m inverse c and you will put x 1 x 2; that is all do the matrix is it clear.

So, what we have converted? What is the, one second order differential equation is converted into two first order differential equations; but they are coupled is not that they are independent not like that form. So, one second order you convert to two first order coupled if there are third order equation, you can convert into three first order equation.

Why all this is done is because, then you have only one solution technique that is it you solve only this problem. If you know the solution technique for this problem, every problem can be handled. Do you understand the ODE? That is why you learnt; now you will be able to relate why you learnt matrix algebra.

See these are very… see initially when it is start - each is start independently then you say I am able to relate various things because if I know, because we are going to have in the aerospace field at least our stability problems, vibration problems, load anything, rotor blade, all problems will come under that. We will try to put it in this; in this type of format and then immediately convert to this format to get the solution.

Now, you know how to get the state-space form. That is why if you are given a differential equation you should be able to convert to state-space form. So, this is a standard that is all you take this take the other term out a m inverse this goes to m inverse k m inverse c straight. Now, we will go and then solve this problem because the solution if a is a constant matrix .

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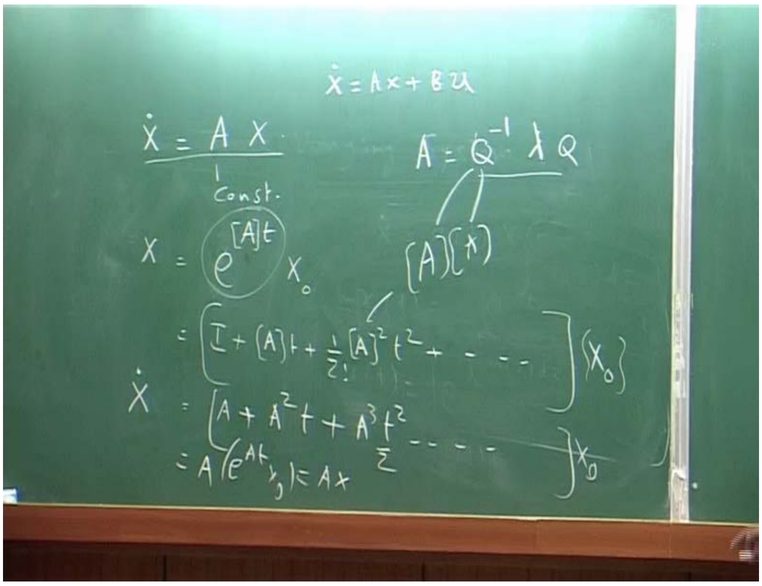

That is x dot is if A is constant then you will say solution can be x naught this is a vector or in other words, you may write it in this fashion: e to the power a t x 0 e to the power a. Please understand, this is a matrix; you can put it if you want, you can put it like this is not that every element is exponentially increased, please understand. This you can expand like I plus that Euler expansion 1 by 2 I think, 2 factorial. What a square right? like that it goes this is the expansion. Now, if I differentiate this into you will put x 0 I differentiate this what will happen? This will be 0 this will be A.

So, if I put x dot this is 0 this will become A plus A what square? t plus A cube t square over 2; like that it will go right into anyway x naught will be there. You take out the a right; when you take out A then what is left is a this is, this A term will be there because if you take a outside this side this is nothing, but 1 plus A t plus the whole thing which is nothing, but the... So, you will have a e to the power a t x naught this is nothing, but A x. So, x dot is A x.

Now, that is why you are doing all these things. Slowly you will realize you can first thing is you learn about diagonalisation of a matrix - real matrix. So, you have the similarity transformation you get the eigen values eigenvectors you pre multiply and then do because when you do that if this matrix you can convert it like the you know that q inverse what I think I will put it A is Q inverse lambda q something like that similarity transformation right yes or no. Somewhere you have learnt, yes this is what. Now, if you substitute this here what will happen is this will become, you can take out q q the inside one will be nothing but e to the power lambda t that is the eigen value.

So, you really know whether the system is stable or unstable or anything like that because a square is what a into a into a what will q q inverse will go up will get lambda square So, we will have a q inverse.

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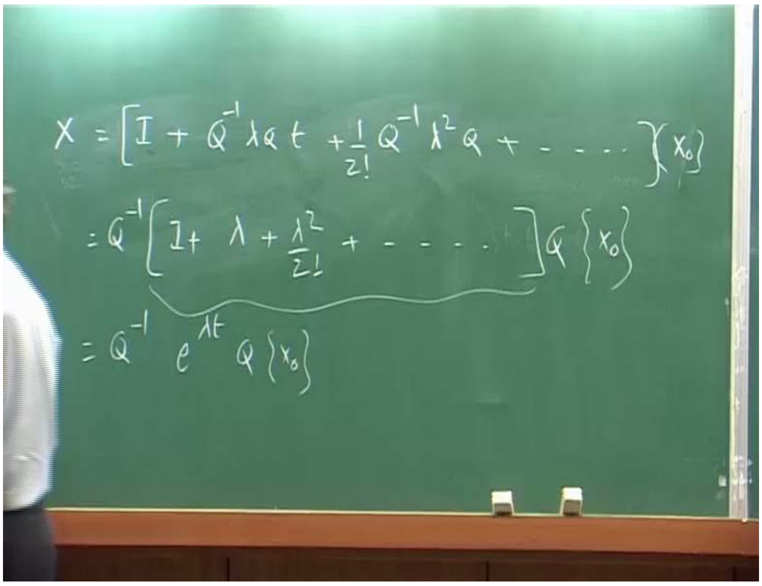

So, I will write it may be that will become see you substitute this here then what will happen x becomes I plus a is Q inverse lambda Q t plus 1 by 2 factorial Q inverse lambda square Q because in between Q inverse Q they will become identity because A square is what A A you substitute this and like that this will go on and you will have x naught what you do is you can write this as Q inverse Q So, this can come as Q inverse 1 plus or I you can take it lambda plus lambda square over into Q x naught this is what.

So, Q inverse a to the power lambda t Q x naught you follow now you see it is a very interesting thing the eigenvector please remember the eigenvectors of the matrix A is the same as the eigenvector of matrix this e to the power A t is a matrix only thing is you have to you do not know how to evaluate that is all because it is a series right the eigenvector of this matrix e to the power A t is same as the eigenvector of A, but eigen values are not same because this is eigtenvalue is lambda for this that eigen value is this e to the power lambda t.

Now, you basically, what you do for this kind of problems if you are given you want to analyze stability because all flight mechanics problems everything you analyze stability you want to know whether system is stable or not you look at straight away go solve the eigen values of this problem once you know the eigen value, it will be a complex because we are not dealing with real symmetric positive definite nothing of that sort because you see this is not the symmetric matrix first of all of course, this is the real matrix this is not symmetric.

So, eigen values can be real complex anything. So, there is no specific condition do you understand? So, you get the routes and those routes essentially tell you how the system is stable or not and if it is stable how much stable or if it is unstable how far it is unstable. In the route locus now you know that why route locus is used and then you want to change the system because, now you say I go and change my system a. So, that makes it stable sometimes you cannot make it stable then you start feedback control. This is all the whole thing starts; the entire subject of then you can have an external loading then the control system if you go they will put x dot equal A x plus B U. Now, for helicopters U is theta naught theta 1 c theta 1 s theta tail rotor.

So, you use the control angle pilot gives then, this now you say you want to stabilize automatic stabilization; that means, you have to go give a feedback based on how a system is measured. That we leave it; you will not bother about the feedback control and other things, just basic system. This is the system; is the system inherently stable or unstable is determinant purely by the matrix A. Here, we look for the eigen value. Eigen values will have the complex routes real part imaginary part.

So, you say your lambda some kth route sigma k plus minus i omega k. If sigma k is positive then you say e to the power positive is, it will grow with time. So, it is unstable;

sigma is negative, it is stable, but this omega will make it oscillatory. Whether it is oscillatory mode or it is a non-oscillatory mode, some modes can be oscillatory some modes need not be oscillatory you can only real routes; that means, this will not be there.

So, this is what you analyze. Now, the question is if a is not constant plus for our if you take the flap equation itself it is a time varying just the flap equation because you can write flap equation in that format. That state-space form; the moment you put in state-space form you will have sin psi cos psi which is basically that is omega t times varying. So, your matrix system, this is time varying if it is time varying how will I solve is a question.

Now, time varying arbitrarily is one another one periodically time varying in the sense a repeats itself in the flap motion, it repeats. Now, that is slightly I do not think we will have time to do that part, but that is very interesting. How do you do that is the Floquet theory. Now, we will only discuss I will just briefly say if a is the function of time how do we get the solution any arbitrarily function of that so, I will erase this whole thing.

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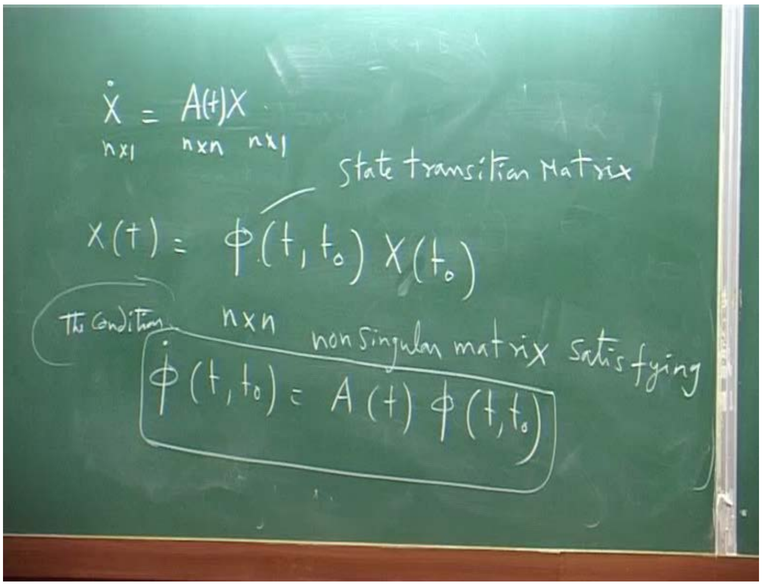

We will say this is a A of t now we can only write the solution this is a matrix may be n by 1 n cross n and n cross 1. I am writing the solution because, we said the initial state in getting change that is all. So, I put this is the initial t naught is the initial condition phi t is the state transition matrix.

This is a n by n non-singular matrix, but what is it satisfying? It satisfies phi dot t comma t 0 is a of t phi t comma t 0; this has to satisfy this condition satisfying the condition you can put it this is the condition please note phi dot this looks like this equation itself, but here these are states this is a state transition matrix.

So, you may tell this as a state transition matrix; now, you can verify why it has to be that is we know that this condition right. The first thing is if you set t equals t naught; that means, this is x t naught x t naught equal to phi t naught t naught that means, phi t naught t naught must be I this is I first condition.

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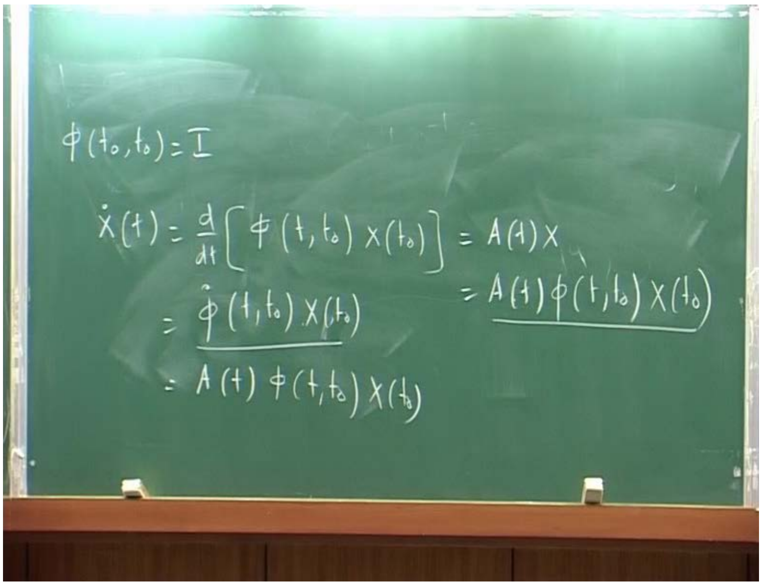

So, you know here phi t naught t naught is I next you take a derivative of this equation because we will give a proof that it has to be like this if you give a proof x dot t is what d over d t of phi t comma t naught x t naught which is phi dot t comma t naught x naught sorry x t naught which is also this is what this is nothing, but our original equation A A of t x this is what this equation is.

Now, what is x x is a t this. So, phi phi comma t naught x t naught. So, you see this is equal to this because either that you substitute; now, in this you can substitute this is what A t phi t t 0. So, phi t comma t 0 x t 0. So, you see x dot t is this x dot t is this which means both are same therefore, the characteristic of the state transition matrix is it must satisfy this condition and phi t 0 t 0 is I now you set the condition, but how do you get it?

There is no easy way to get it; you cannot write it as a matrix exponential. Why we cannot write it as a like we wrote e power A t in the single case we put integral t 1 to some t 0 to t a f tau b tau can we do the same thing here you can do provided and show condition is satisfied otherwise you cannot do is this clear I will erase this part. So, you are given the proof that phi dot, what is the condition for that transition matrix it must satisfy this with this, that is all.

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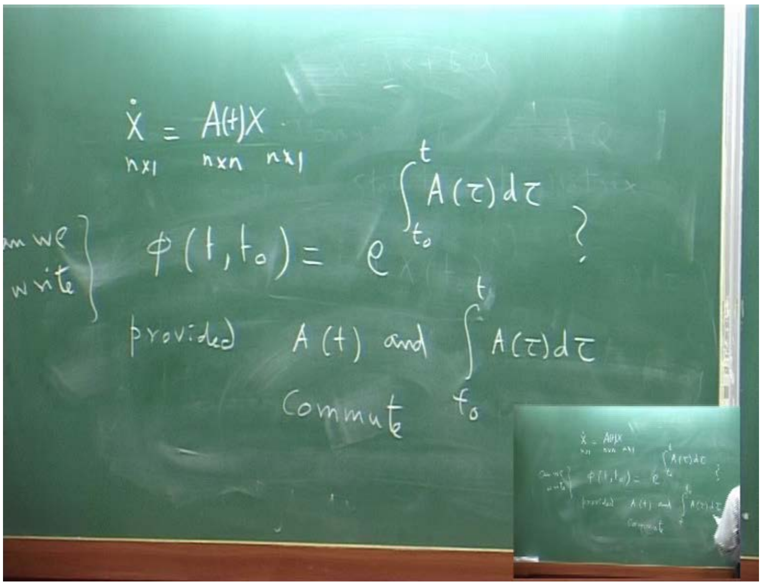

Now, can we write phi as a matrix exponential? that is I am going to put as you said this is the equation phi t t naught I am going to write can we write e to the power because what was our equation? Our equation is phi t t naught dot equal A A t phi t t naught can we write in this fashion? You can write provided that a f t and integral commute please note provided 80 and this commute - commute means this time this is equal to that time;this a b equal to b a only if they commute otherwise you cannot do it.

But it is possible only when A is constant or if A is diagonal otherwise you cannot do it for a general case you not say A B equal to B A please note that that in the matrix 2 into 3 number yes 2 3 is 3 2 that is 6 this is 6, but you cannot put it for matrices. So, these are all very important that is why as on operation is it commute. So, provided it commutes you can write the solutions suppose if they do not commute sorry this is not the solution then how do you say they have to commute? That is purely from the this proof is here I will give you proof that is you take you do them expansion this e to the power this is a matrix only.

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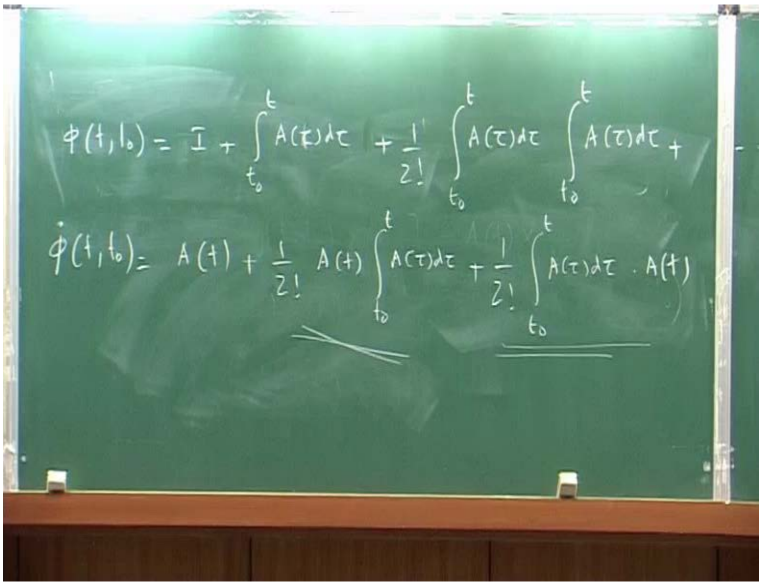

So, you can expand it expand it I plus integral t 0 to phi a f t sorry a f tau d tau plus 2 factorial you will have 2 t 0 t a tau d tau a tau b tau plus, so on, so on, so on Now, you see this is the expansion now if I differentiate phi dot phi dot this is this is A t, but this will be 1 by 2 factorial A of t integral t 0 t A tau d tau plus 1 by 2 factorial integral t 0 t A tau B tau and then A t because this first you take this thing now if these two commuting then this is you can take out a outside if they do not you can write this solution you follow.

Now, this is the main problem, if they do not commute, how will you write the solution? If they commute you can put it in this format. So, do not commute; I will write the solution. It is no way you can do by hand; it is a you have to numerically calculate or some simple problem you can integrate them. So, what I will do? I will show that part if they do not commute the general solution this is again a series.

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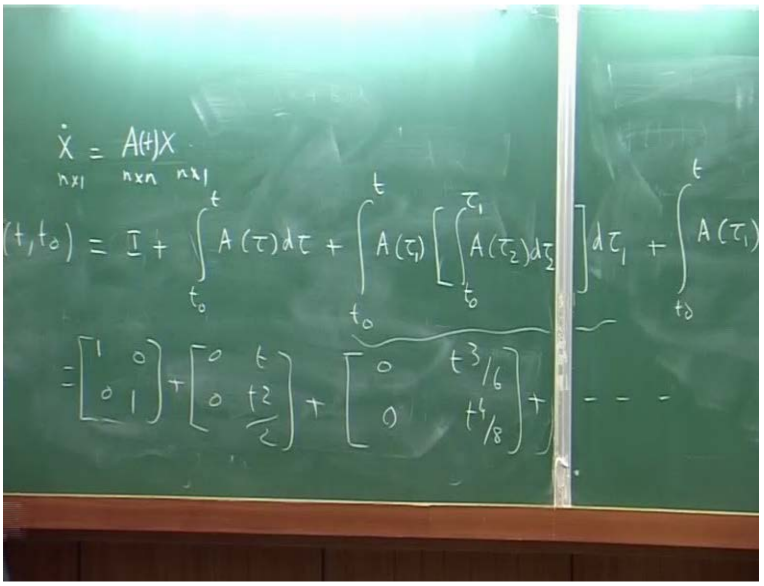

So, general solution is this is a series expansion: I plus integral t 0 to t a of tau b tau plus I will put tau 1 a tau 1 I open a bracket here I put one more t 0 to tau 1 a tau 2 d tau 2 close put d tau 1 please note this is a integration inside you do. And, if you want to go third term it will be still t 0 to t A tau 1, you open a bracket t 0 to tau 1 A tau 2 open one more bracket put integral t 0 tau 2 A tau 3 d tau 3 close d tau 2 then close d tau 1 plus. So, on. So, on. So, on this you can show because what you do when I differentiate this phi dot this is 0 this is a t here what will happen because the entire thing is over. So, tau 1 this will become a t and this will stay as it is because this is just dummy index like that what will happen is 1 by 1 it will stay as it is and this is the c d solution we have to do numerically only you cannot there is no way; you can compute this easily this is as far as a general a of t is concerned.

Now, I will give you one small example. I will solve after that we will, I will just briefly mention, but we will not be able to do it because that derivation takes lot of time that proof, but that is an interesting thing.

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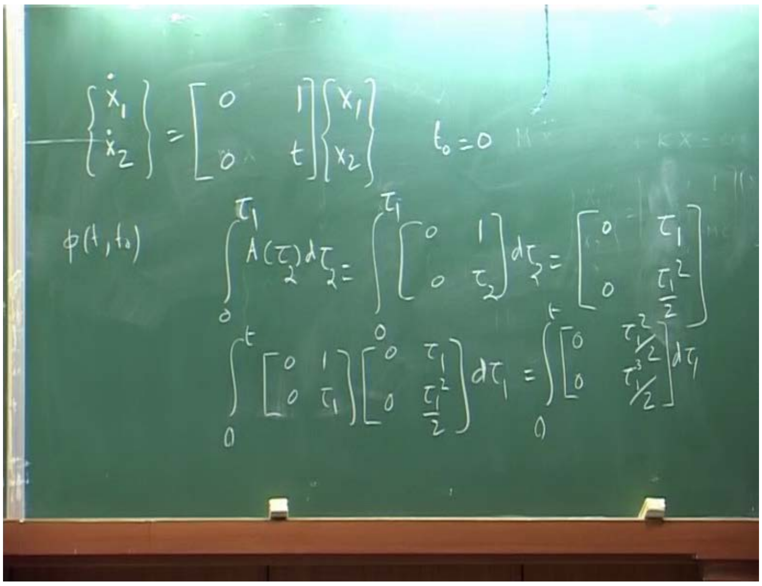

Suppose you take a simple problem dot dot 0 1 0 t; now you try to get the equation by yourself with t t 0 as 0 compute transition matrix transition matrix bearingly what is the solution to this problem. So, you solve phi t comma t 0. So, what it will have very first is I I is as it is then the second term will come A tau d tau you have to put that; that means, 0 1 0 t put the integral that will become 0 tau.

So, this will become first terms that is integral 0 to t A tau d tau is nothing, but 0 to t 0 1 0 tau d tau which is that is a first part first term this is the first term now you have to go to the second term. So, this is I maybe I will write it this is 1 0 0 1 and this term is 0 t 0 t square over 2 then plus we have to calculate this term that will be because we can use tau maybe I will do it; that term will be 0 to first evaluate tau 1.

Now, this will be, you are going to have this is tau 1 tau 1 square over 2. This you multiply to now you have to do this 0 to t A tau 1 A tau 1 will be 0 1 0 tau 1 multiply to 0 tau 1 0 tau 1 square over 2 d tau 1 because this you put then you multiply when you that then you take that matrix multiplication this will become what integral 0 to t I think 0 0 I think tau 1 square over 2 and then tau 1 cube over 2 d tau 1.

Now, this again 0 0 this is tau 1 cube by 6 and the limit this is tau 1 4 by 8. So, that will be t. So, the third term will become like this 0 t cube over 6 0 t 4 over 8 plus so and so forth because, I am not this is what the general solution is.

Now, you add all of them; that is your solution. So, 1 this term will be 1 here you will have t t cube by 6, etcetera. Here you will have 1 plus t square over 2 plus t 4 over 8 etcetera. So, like that you will have a long series this you have to do only numerically. Now, the next question comes you quote a very general A t. I am not interested in a t I am interested in A t which is periodic, but there are very special equation. This you may be I do not know whether you are exposed to or not there is something called a Mathew equation Mathew.

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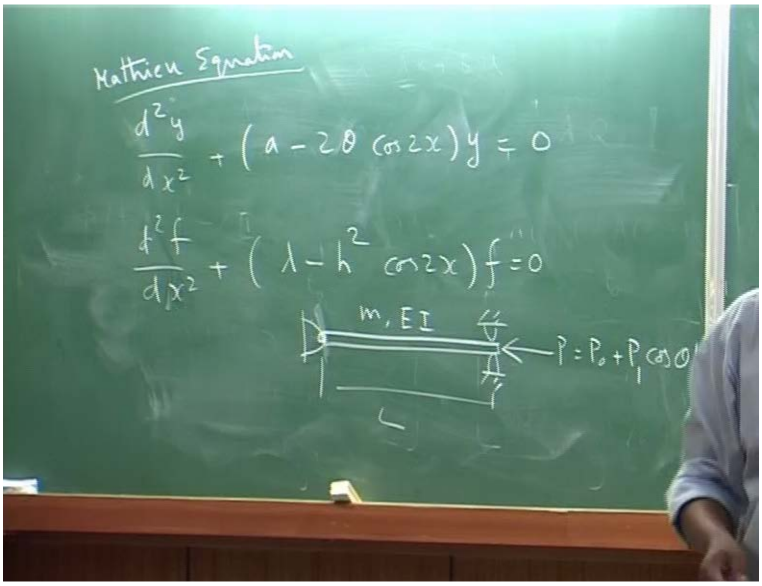

It is by, I will just write that equation this is just for your this is Mathew equation this is one or you can write it in another form also there are various types of ways it can be written another way is the write d square f by d t square sorry d x square lambda minus h square cos 2 x f 0 any form maybe all are pretty much same this is called Mathew equation here it is periodic.

But of course, this is the second order you have to convert to first order you can do what initially people are interested in seeing for what combination of lambda and h square combination the solution is stable. So, if you look at some advanced books they call it some c function s function something like that though will be there; they will give the condition of stable condition; that means, at this for this combination of values I will get a stable… stable means a periodic solution and there is a diagram; I cannot draw the diagram.

The region of stability for what combinations it will be stable; for what combination the system is unstable, but it will not tell you what is the value of damping you can only say stable, it is unstable. now you may ask how stable it is am I near the imaginary axis how close I am. see if you are very far on either say you are highly stable, but if you are very near your stability is just a marginal stable.

That result will not give you that, but it will tell you the regions; that is the separate formulation derivation you assume a series solution and then put the condition it is stable then what should be the values there are various the period of the solution can be t or t by two you know various types of sin function, cos function, all those things are there; I leave out that part.

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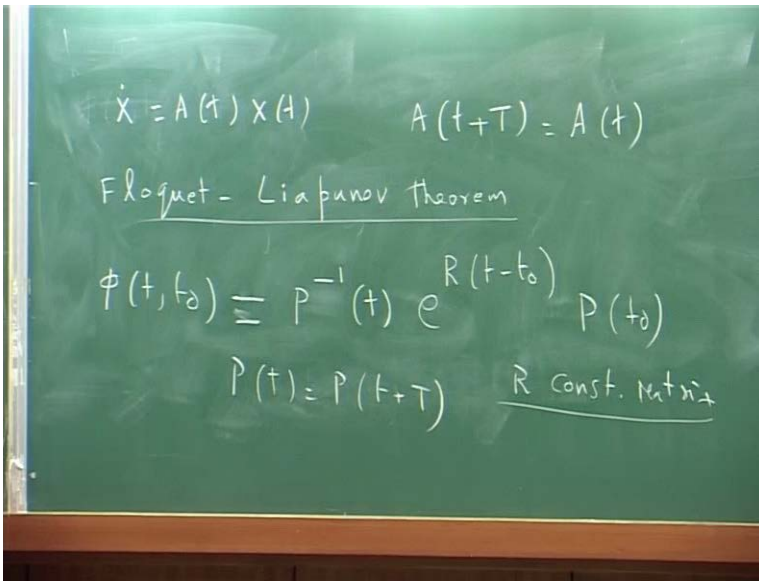

The (( )) theory says if x dot is A of t x t and A of t plus t is a of t periodic because, we are interested in periodic solution. If I will not give the proof, the proof I have; I will simple mention this statement if you have this condition, what is the form of, please understand, by transition matrix not a form, this is my equation which is periodic please understand.

This is a time varying equation. Only thing is time varying is rather than very general I wrote that long; what is that? That series solution if A of t is periodic, the Floquet theory; this is what Floquet we call it Floquet Liapunov theorem. Floquet theorem - anything you can tell Liapunov you may find it in different form, but I have it from some notes I am keeping this way.

What it says is the form of this; there is a proof I have it will be in this form and p of t is p of t plus the form of the transition matrix is this the proof is here. I will not go into the proof because, it takes about 2 3 pages of notes, but it is a very interesting proof. Now, you know this is the transition matrix. You try to find out basically r is a constant matrix r is constant matrix. The whole purpose is finding out the transition matrix. First, how r is related to the, I will tell you, what r is related? That is at the end of the proof it will give you; this R is a constant matrix which is given by e to the power R T is phi t comma 0 e to the power capital R into T 1 period.

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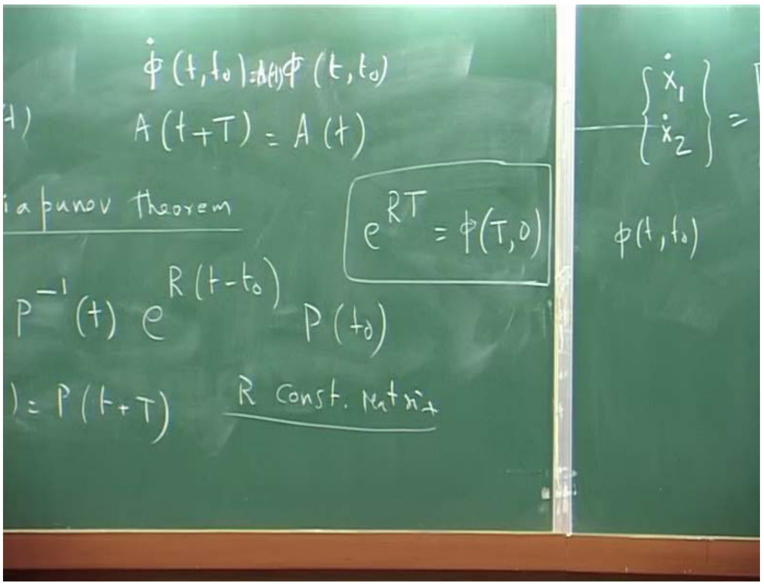

So, what is done is, you start at 0 the end of one period. What is the transition matrix you evaluate because you are given the equation all transition matrix have to be, have to satisfy what this equation right sorry a t phi t comma t 0. They have to satisfy this equation; always you try to calculate this matrix. By solving this differential equation at the end of one period, you calculate. Once you calculate you know the relation. You find out the eigen values of this. This eigen value is related to this eigen value. Immediately you will know the eigen values of R; what is the real and complex. If it is stable, you will say my system is stable; if it is not my system is unstable.

So, this is the procedure that is adopted to find out whether stability of the rotor blade in forward flight; not for simple flap problem. Flap lag torsion stability they use the Floquet theory get the transition matrix and then roots of that. Theoretically, these are all done; theoretical calculations industry whether they use it normally they do not bother. I will put it away because you try to design the system size that you do not get into any of these problems. This has an interesting result; you want to know what is the damping that is available; you want to know what is the frequency, what is that then? Academic side yes it is very important, but one of the equation I wrote you, this equation right? The problem, this is not just arbitrary. There are several examples for this. One of the example because you are all structures people, one of the problem is this I think it is a some cant beam with the… it is a buckling problem.

You all know Euler buckling static problem but, only thing is this tip load is time varying. If you write the equation for this in fundamental mode because this is a continuous system you have to do single mode. You do finally, it will come to this form; it will come to this form in one mode. Each mode it will be the same equation. First mode, second mode, all modes will then they start looking at whether stability of this for what values of lambda x they are related to the this properties and then p 1 how they vary.

Whether the system will be stable or unstable; unstable means, it will just vibrate and then break. So, if you will go to Google search Mathew equation. They will give various places, but for structures from the point of view of vibration and this is the problem. Suppose I give you this problem, go and derive the equation. This is an axially loaded beam if you want to derive this is of course much easier because I am putting same axial load throughout. I can vary the axial load; this you study if you take theory of vibration next time.

The procedure is get the transition matrix, relate the eigen value. I am not going to the detail because little advanced to this and stability of the system is purely based on the eigen value of this transition matrix because composite materials for rotor blade. You always mention that it has a good of course, fitting characteristic is good; damage tolerance is good and you can tailor the properties. Tailor means you can create coupling bending torsion coupling anything by keeping the fiber.

But you do not do any such things in the manufacturing of a actual rotor blade because, you still do not know what all this coupling effects will really introduce to the blade dynamics. So, the composite material is used in such a way the layup is that the final design will look like an isotropic plane. So that I am safe do not try to put too many things and then finally, you will really have no clue what is happening because whether it is due to aero-elastic problems or whether it is due to this (( )); so, most of the times industry uses the material for something.

But, tailoring even though it is said make sure that you do not get into any of these problem unless you are very thorough about what happens because, very one interesting point I write somewhere they replace that actual blade metal blade. See, this is one of the projects you say I want to replace the metal blade by composite blade. You know the dimension, you know the mass distribution, you just make a composite blade this is a major project; that means, all my helicopters I will throw away, old metal blades; now I will use composite blades.

So, one of the project they change; what is that everything should be? You know dimension should be same; you know the mass distribution should be same; you want the stiffness also to be distribution to be the same and then only you will say dynamically they are same and they made a blade because dynamically means how many modes you will make it? First mode, flap mode, lag mode, torsion, few of them you say metal blade. Whatever it has I will have the same thing in the composite blade design. It is not that easy design; you made the blade put it on the helicopter; you fly you find lot of vibrations.

This was in one of those articles; why question mark? That is all that is the end because nobody knows what is going on and I say that everything is same but, I put this. I have more vibrations you follow what I am saying? So, there are many things which we do not know and therefore, but you take the risk, but very calculated risk. That is why if you listen to the talk what that visitor Vishwanathan - Doctor Vishwanathan mentioned it is very important; but you try to learn more but, always cautious in your design conservative thing because you do not know several things.