Instability and Transition of Fluid Flows

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Module No # 01

Lecture No # 06

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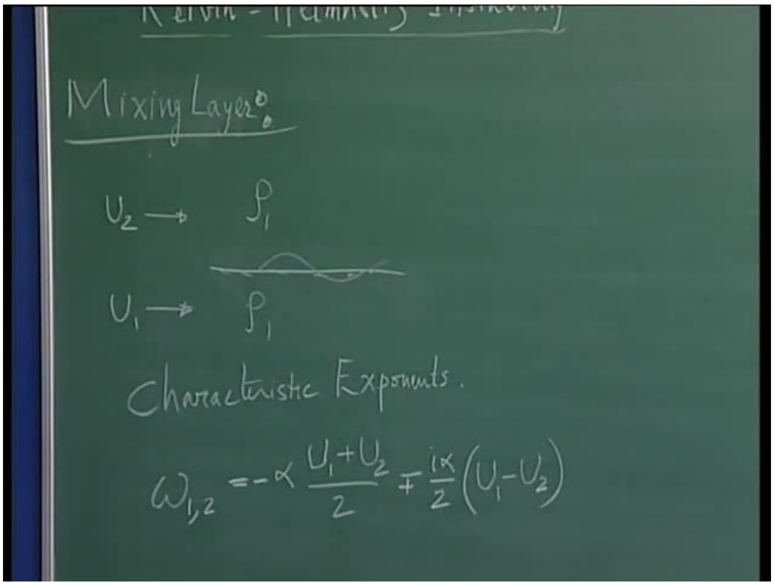

In the last class, we were talking about Kelvin-Helmholtz instability, and one last topic that we wanted to discuss was the mix problem of the mixing layer. So, we have the same fluid rho 1 and rho 2, but it is approaching the interface of two different velocities. We worked out the characteristic exponent like this, and because there is this part with negative imaginary part, you can immediately see that it is unstable, and the second thing that you also notice that this omega does not depend on theta.

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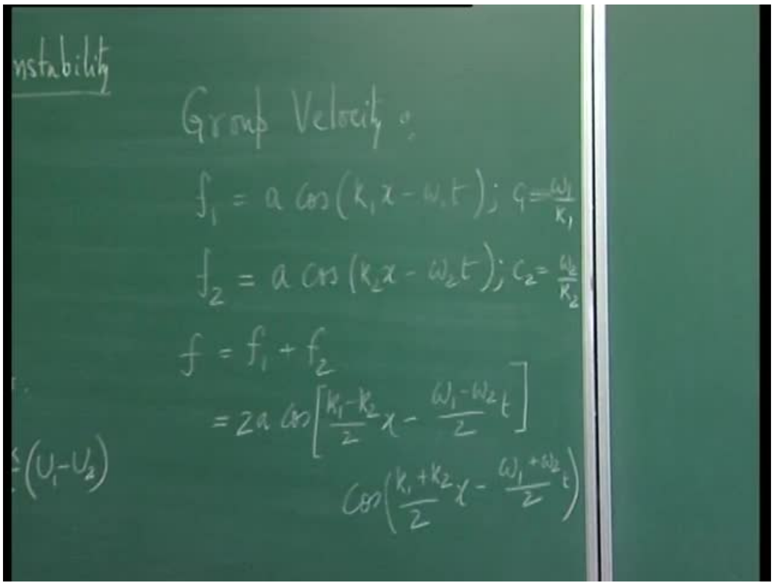

So, whatever may be happening in the y directions not reflected in the time variation, so that means that the flow behavior is totally dictated upon by stream voice variation .And without spending too much of time that when we have a collection of waves as we have in this case because we are integrating overall possible alphas. How do those alphas react interact with each other, that concept is brought about by the concept of group velocity. So to demonstrate this group velocity idea, let us say we have two functions f 1 a cos of k 2 x minus omega 2 t, and let us look at a similarly another function, which almost have a same amplitude. We will take it as a x out then we will get it like this, and now if this two waves are simultaneously present as you can understand this is a wave, the wave propagates with the phase feed c 1.Here that is equal to omega 1 by k 1 and for this wave c 2 will be omega by k 2.

So, this is the speed at which the phase changes. How does the energy change that is what we are interested. So interaction of these two waves; if I write it as a f one plus f two, then we can see that this is a simply equal two a cosine k one minus k two by two x minus omega one minus omega two by two t whole multiplied by cosine k one plus k two by two x minus omega one plus omega two by two t. So this is what you get.

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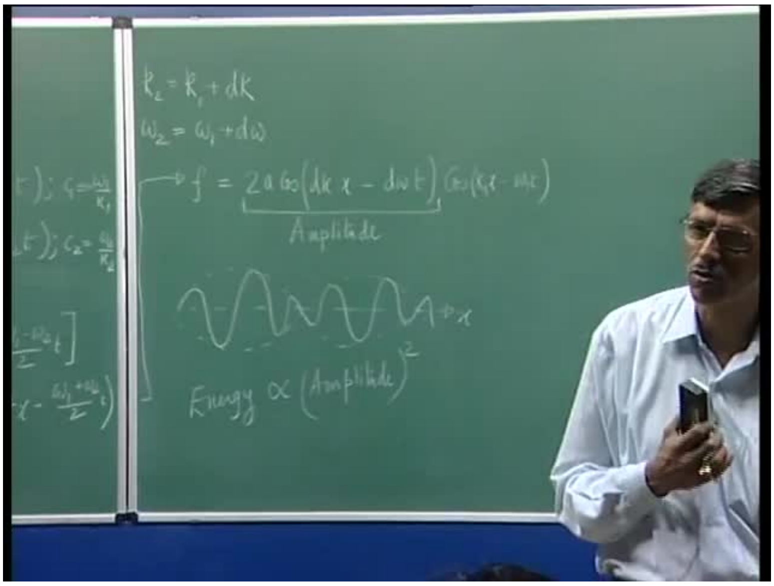

Supposedly, say k 1, k 2 are neighboring elements in the wave number space, that is if we write; k 2 is a incrementally different from k 1 which I can write like this. We now know what a dispersion relation is; dispersion relation is a relationship between omega and k like what we have shown here. This is a dispersion relation; k is replaced here with alpha, so that is what you are seeing. Dispersion relation relates the time variation with the space variation but not in the physical plane but in specter plane. So, if I have a dispersion relation and if I focus in the attention onto neighboring wave numbers like k 1 and k 2, then the corresponding circular frequency would also be incrementally different; it would be incrementally different.

Then from here what we are going to see is f is nothing but 2 a cos d k times x minus d omega times t, and then what about this, this will be like an average. So that is like, method variation will be like k 1 or k 2times, so I could just simply write it as cos k 1 x minus omega 1 t. Now what you are noticing that, there is a slow variation that is indicated by this part, and there is the usual variation that (( )) apparent constituents. The apparent f 1 and f 2 varies like this.

So, what is you are seeing is that two neighboring waves when they are simultaneously present their net effect is to cause the usual variation that each constituent has, plus an amplitude which slowly varies and this is something what you see if you plotted at for a fixed time. If I plotted versus x, then I would see that the variation would be given in a packet like this. So, the actual way in that you would see would be something like this, and so on so forth. So what is happening, you can see that the original waves which were very close to each other their amplitude actually varies between plus or minus 2 a. So, the individual ones are going to vary plus or minus a, but their combination varies from plus minus 2 a. Basically, this is something like you are slowly varying amplitude, and this idea was brought to us by actually Hamilton and later on it was refine by Raleigh. What we notice is that these two signals do get modulate, this is what we call as a modulation; two constituents getting modulated and their amplitudes are travelling twice the individual constituents. What happens is the energy is proportional to amplitudes square, so if the amplitude varies by this law the energy also varies according to this law.

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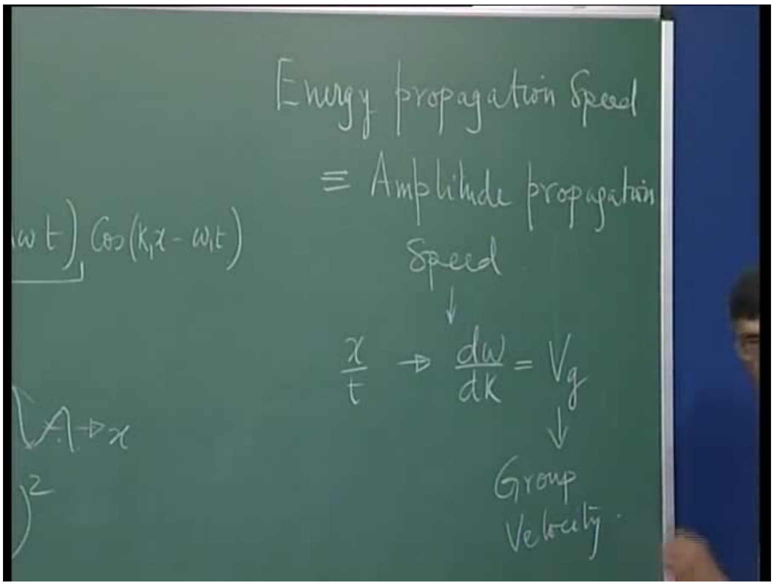

So what we are saying is that the energy is proportional to amplitude square. So the speed at which the amplitude propagates that would also be the speed at which the energy propagates. Basically, we conclude that the energy propagation speed could be equal to amplitude propagation speed, and what the amplitude propagation speed is that you can see the way it is. So what is this, this is going to be the speed at which x by t line moves, and what is that that is what I can see is that constant phase of this amplitude would give me this. So this quantity is given a name which we will call as v g, and this is what is called as group velocity.

So what we are getting is a new concept, that if you have a collection of waves then around the center of some wave number we can see they all group together, and they have this constructive interference with each other so much that the amplitude actually doubles up in some place. The place where it doubles up are called the antinodes and where it is zero are called the nodes, and those propagate those group of waves or wave packets propagate with group velocity. So, the group velocity is given by this d omega d k. So this k that we talk about basically the wave number vector could go in any direction. What we were studying in the context of Kelvin-Helmholtz instability is that k had two components, the x component was alpha, and the y component was beta. So I would also have a group velocity as a vector, and the individual components will be given by the corresponding relation that we have talking about. It should be written as d omega d alpha, that will be my group velocity in the x direction, and if I could get d omega d beta, that will be my group velocity in the y direction.

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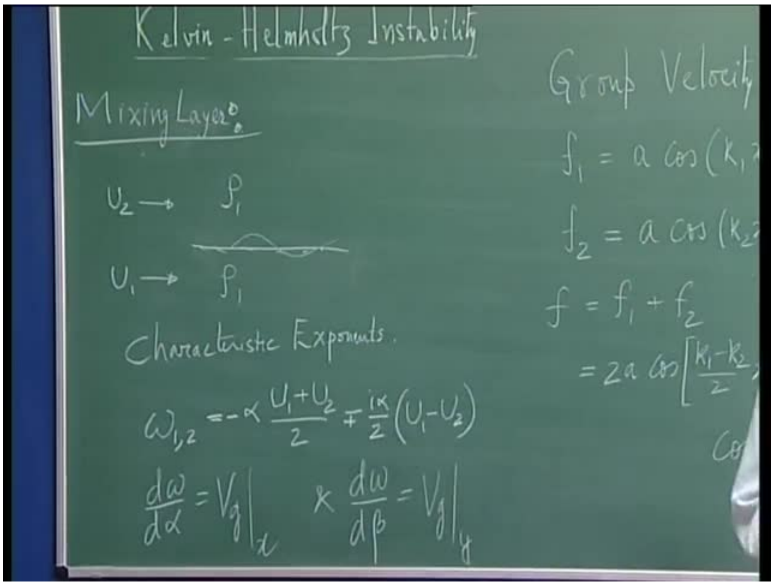

So, I just gently introduced you the concept of velocity to appreciate what happened in the case of Kelvin-Helmholtz instability. You can very clearly see the omega as a function of alpha, but omega is not a function of beta. So what happens, if I create a disturbance the energy will always propagate in the x direction and not in the y direction. This is something that we need to understand, and close our discussion on Kelvin-Helmholtz instability.

So we have been gradually increasing the level of difficulty or the variation of the equilibrium flow. First, we studied the equilibrium flow was zero at dynamic stability of atmosphere, now we have studied a case where we have the equilibrium flow given by two sets of uniform flow. Now, let us go over to a engineering problem which we (( )) except really a problem not only of engineering, but it is also problem of physics which is interesting enough. I have exercise the mind of great physicist of starting from Kelvin-Helmholtz and Raleigh, you can name many faces to our not our really double with it Now let us look at the problem where people started looking at different idea altogether, ok. Now the idea is what happens in a real flow. Unlike, what we suggested that this could be trending edge of an aero fall and we could get this. But you still realize that it would not be uniform flow above and below, it will have some kind of a distribution with the normal direction.

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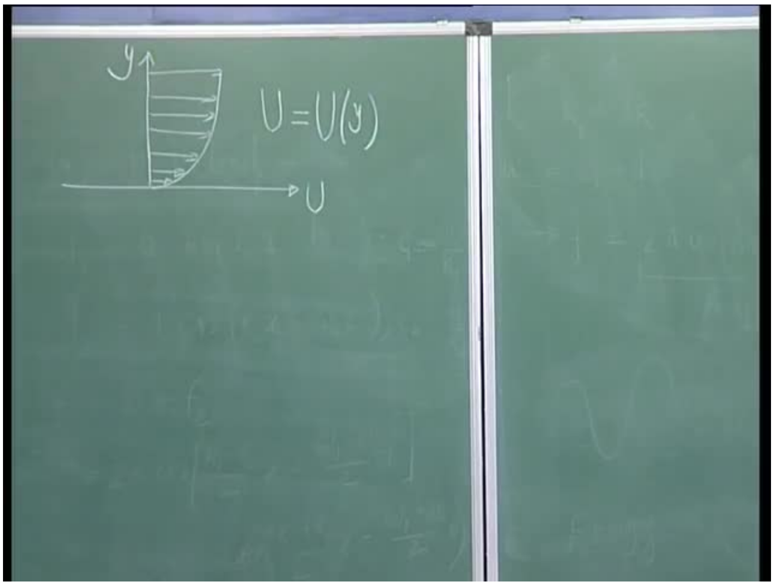

Basically, we can study the flow field which has this detailed variation of the velocity profile, given like this. So at different height I am going to get this kind of velocity distribution, and what is the characteristic of this velocity distribution that it maintains a shear. So the mean flow now has a shear, whereas what we studied before; zero shear case. So that is what I said that we are going to retch it up the difficulty of the description of the equilibrium flow.

Now, we are talking about a flow which has this kind of (( )), for the sake of convenience we will always represent the mean flow with capital letter and we keep the lower case notations for disturbance quantities. So what we have here is, of course the mean flow earlier is just simply, U2 is constant, U1 is constant and here U will be u of y. So people will be aware of it. Raleigh Helmholtz Kelvin they were all aware of it that the mean flow can have shear.

Now the question is when you have shear what happens to the disturbance field. This is where actually philosophical decision was made. This is something what you call as a paradigm. The paradigm at that point in mid nineteenth century was the following: the shear is caused due to what; viscous action. So, viscous action that gives rise to shear what it can do to the disturbance field, this can attenuate. That is a very commonsensical clue that one would like to drop, if there is some viscous action it is going to attenuate the disturbance field.

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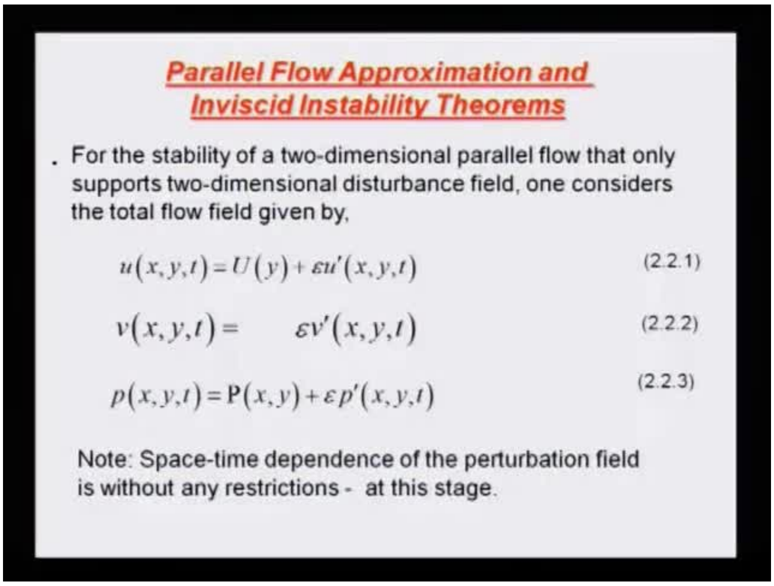

So making that observation all this joins physics. What they decided to study is, instead of studying the viscous stability they started studying the inviscid instability. So what happens, what does it mean, pay attention to the flow field description. Let us say, we are talking about a two dimensional flow, so we will have only two components of velocity U and V, and this as associated pressure field; we are talking about this two dimensional field. And also do understand whenever you study disturbance, you study it in original form. So we do not presume it to be study or on study.

So that is why we actually talk about the total velocity field and the pressure field to be function of time, while the equilibrium flow could be time independent. But the disturbance quantities have embedded in them all possible kinds of space and time dependent, so that is what we have talked about for the ease of our analysis. Note that we have looked at the velocity profile which has only single component that is in the stream of Y direction; so U of y. That is what we have, and this also tells us that this does not vary with x. That means, this mean flow or the equilibrium flow that we talk about is only a function of the wall number coordinate, but it does not vary as we go along.

So what happens, what kind of flow is that, that sort of flow is called parallel flow, why because everywhere it is same. If I try to think up of stream line what will it be for the same height at two differentiations, it will be the same value because it is only function of y. So what happens, stream line happens to be parallel, that is why it is called a parallel flow. So what are we sacrificing in making that assumption, we are sacrificing the growth of viscous action. What is the viscous action; the distance of a boundary layer let us say. So in this frame work when we are saying there is only U of y and V is absent, we are talking about stream lines which are parallel to the wall, we are calling them parallel flows and we are neglecting the growth of the associated shear.

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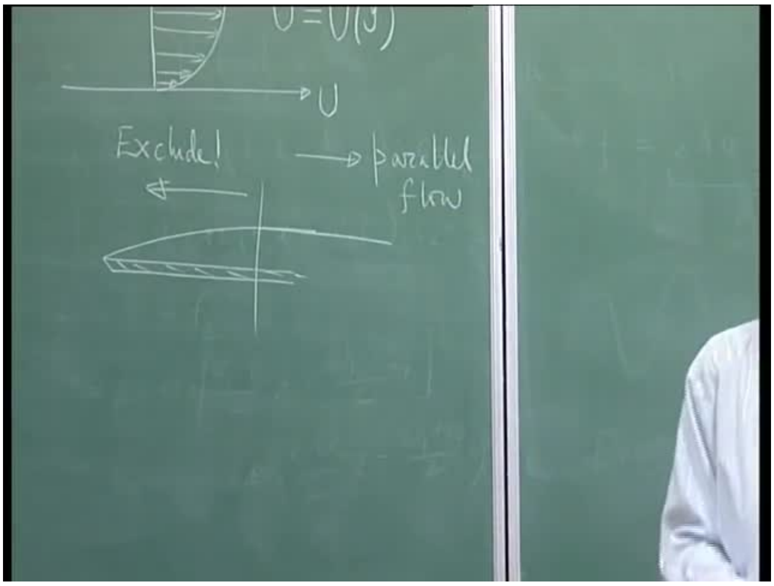

So, with that we will begin our discussion, understand what a parallel flow approximation is; it is an approximation for a flat plate. If you are looking at, if this is a flat plate with a sharp letting edge then what we are going to find is that, the boundary layer would form which you have a consideration like this if we are looking at a consideration like this. What we are saying, in making the parallel flow approximation that picture is not consistent with this path, where actually boundary layer is growing from zero to some substantial value and then subsequently it could remain; that is our constant. That is precisely what you have really do get to see, even if you go to a tunnel (( )) experiment we are going to get a similar profile.

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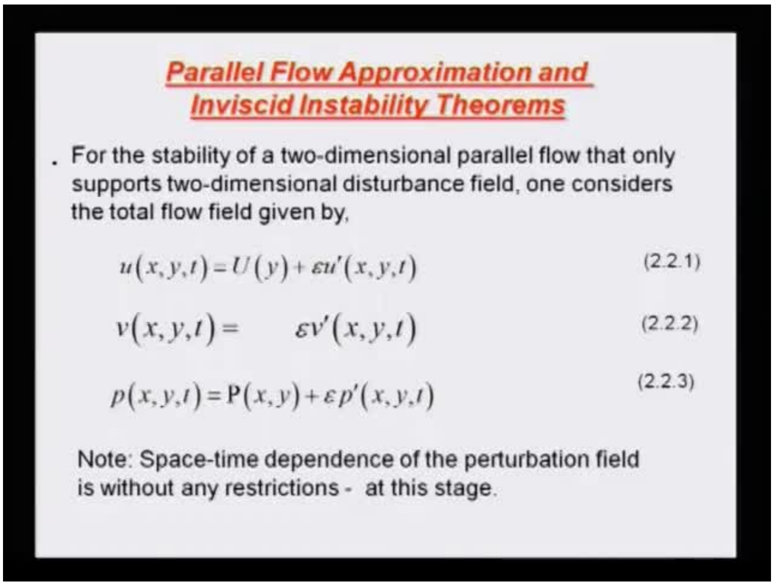

So, if I leave out this part of the flow, we exclude when studying this and this is where we are kind of justified in talking about parallel flow. Then we also start with the paradigm of inviscid studies, let us see what happens. So, this is the way that we are going to talk about, and I specifically note here that the space-time dependence on the perturbation field has not been restricted, while the mean flow has been restricted here. So what do we do, we try to write down the governing equation.

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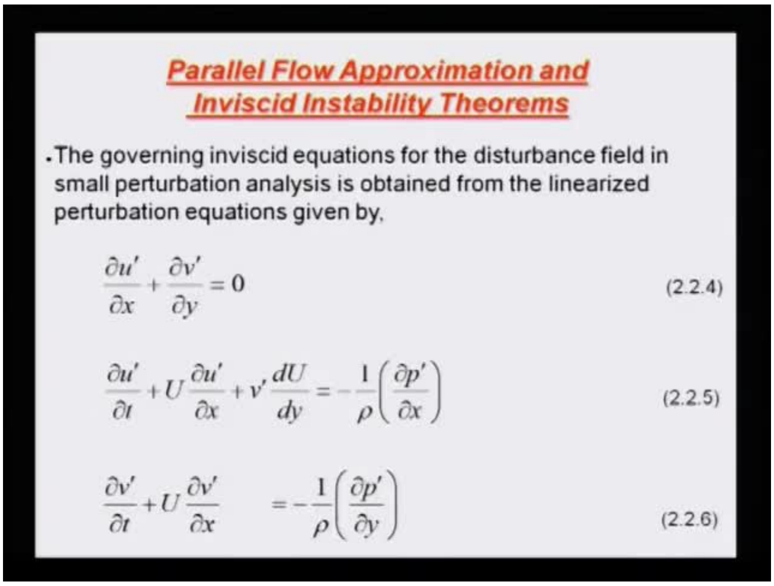

What have we done, we have taken the navies stokes equation there and then we use that splitting that we have shown. What was the splitting, the mean part plus a small perturbation part which was given in terms of that epsilon parameterizes the smallest of the disturbance field.

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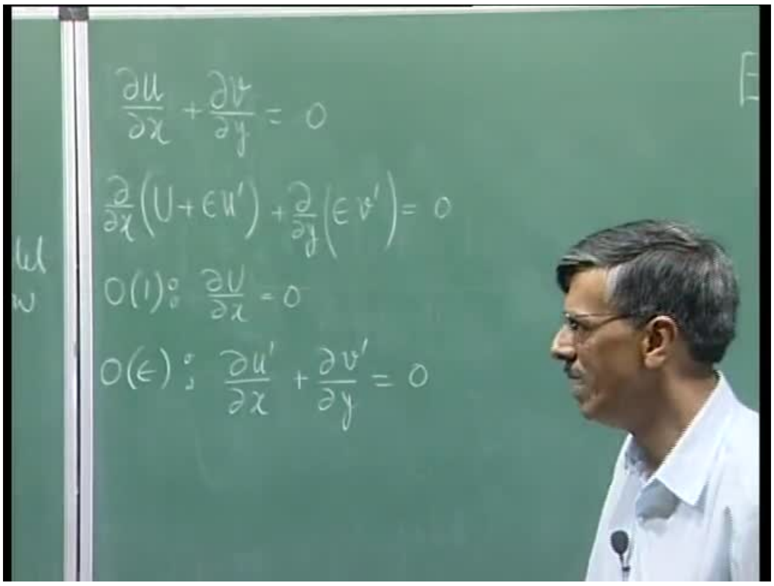

So let us go ahead and see what we get. Our full navier stokes equation for example is given by this del u del x. We are talking about incompressible flow, and in a cartesian frame that is your equation. Now what is our U, it is nothing but the mean flow plus the perturbation field. What about mean, that does not have a mean component, we are talking about parallel flow. So the parallel flow (( )) having mean component so that is not there then we have this, and you can see the usual formal way of doing perturbation series is (( )) write down one quantities and order epsilon quantities. What is the order one quantity, it is only here. This is nothing but your restatement of what you are really assumed, approximated that U is not a function of x, it is a function of y alone. So, there is no contradiction here, the basic reaffirmation of what we have started with and that is what we get. What about the order epsilon on quantity, it is one comes from here and one comes there. I have just factor out the epsilon, it is multiplying each wall of the term.

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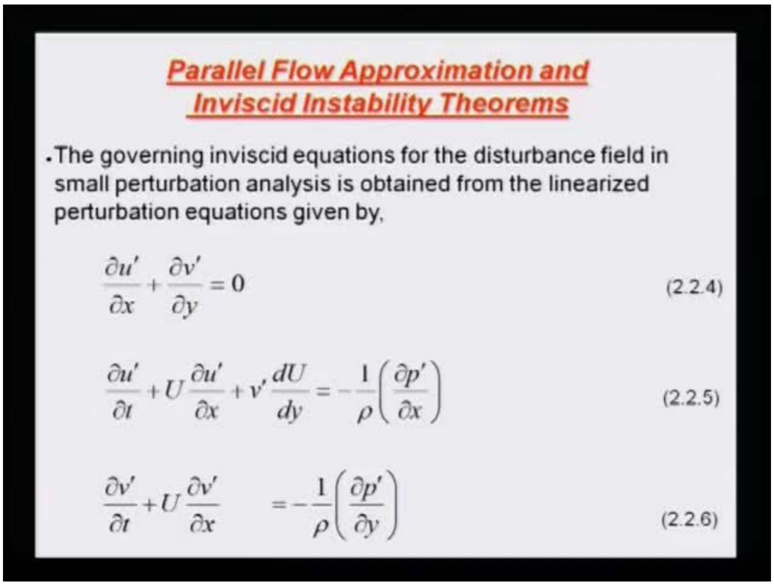

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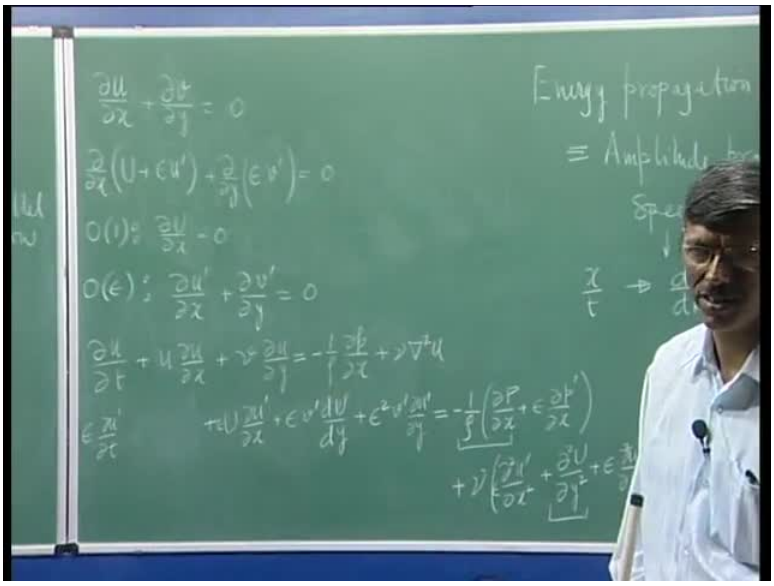

That is what you see the slide there, the mass conservation equation gives you this. So the perturbation quantities satisfy the mass conservation in this form that is what we have derived here. You can similarly go ahead and look at, let us say the x momentum equation that is this del u del t plus u del u del x plus v del u del y, there we will have minus one over rho del p del x, and since we are studying the inviscid instability what we are going to do we are going to write this equation alright. Now you have seen what about U is this plus that, what about its time variation yes looking at the stability of, the stability sort of a study flow. If you are doing that we will not get anything there, what we are going to get epsilon del u prime del t. What are we going to get from here, we are going to get a mean term it will be u del u del x that is not there. del u del x is zero we have already seen.

So this terms does not survive, what about the other term, they also get U del u prime del x hum, and we will not have the other term u prime into del u del x because del u del x is zero. So, we do not need to really write this term out, so this is not there. So we do not have that so that is done. What about this term, what do we get from here, we are writing out order one term does not exist; order epsilon term will be epsilon v prime. epsilon v prime. What about this, we will get only the mean flow term coming, that is our order epsilon term.

So, that is your del u prime del u del y that is this. In fact, what you need to do is since you have assume u to be function of x only, do not even write to need to write this as a partial derivative you just simply write it as the alternative derivative. What about additional tau, there is additional tau epsilon square that is v prime del u prime del y, so that we have written in up to whatever is there although hand side density remains invariance. So we have no problem, what about the pressure; I have a pressure term which I may write del p del x plus epsilon del p prime del x. What about terms coming from here; nu the x derivative, what will I get for the mean flow it is not a function of x, so there I will not get anything but I will get this kind of terms and that is multiplied by a epsilon.

What about the y derivative terms; the capital U is a function of y, so we are going to get this plus epsilon del square u prime del y square. So we have written down everything now, see what we are doing, we are looking at order one quantity, so what happens the order one quantities are here and here. So what does it tell you, this is like your channel flow equation, you need a pressure gradient to overcome the viscous resistance, but the pipe flow and channel flow, you have to spend some energy to make the flow be there and that is what these two terms balance each other.

So, if we are writing now the order epsilon quantity I would not keep this term. Now, if we are writing order epsilon quantity, we do not need to even write the order epsilon square term that is also not consistent, so what we have is this set of terms. Now everywhere epsilon is there, we do not need to write your epsilon because you just simply get rid of them and we will have this equation. This is the paradigm that we talked about come about what is the paradigm that the risk of action of it is there, it is going to attenuate and make the problem somewhat little more difficult to handle. So you understand this is the way actually the early part of fluid mechanics developed. You have the key if you look for the lock.

So people went about the other, they will first get some solution and then see where it can be used. This is a similar approach here because we cannot handle the full thing and then as after, thought we say viscous action is there it has to attenuate. So, let us try to find out the more critical case by dropping the viscous term altogether that is essence of inviscid instability. Inviscid instability would mean that we do not need to look at this term and that is what you are seeing here, two point five is a same equation we have derived it. Now however, derive the y momentum equation, you can do it yourself and convince at, this is the equation that you want to do.

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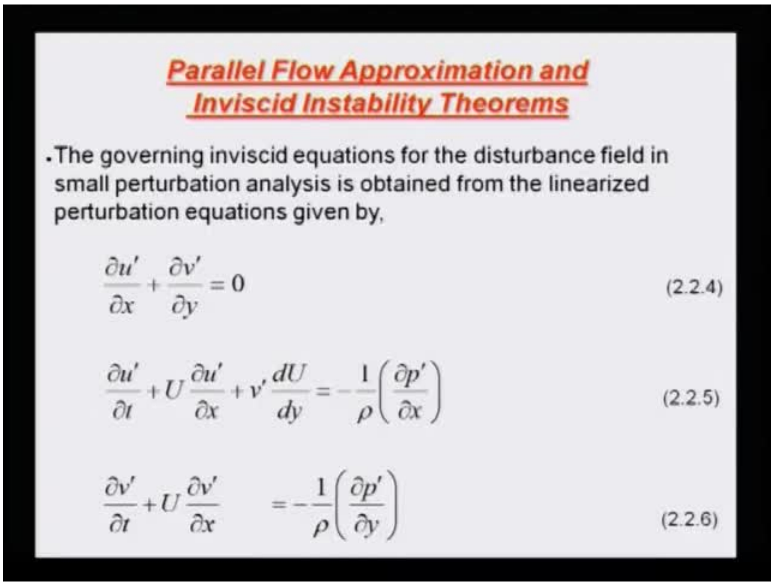

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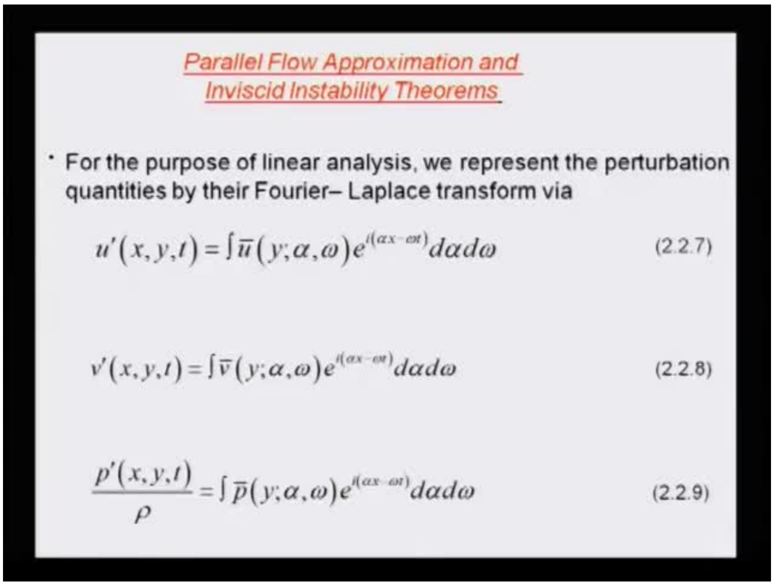

Basically, we have now a situation where we have obtain the governing equation for the disturbance quantities governing equation for the disturbance quantities. Now what we are going to do is the same thing that we did in studying Kelvin-Helmholtz instability, we want to look for more accuracy. There may be more tangible means of analyzing it. So we represent this disturbance quantities in terms of fourier Laplace transform. So for example, U prime; I will write it as a fourier Laplace amplitude that we have indicated by U bar, and then the phase path e to the power i alpha x minus omega t, and they have to be integrated over all possible alpha and omega. So it is basically double integral. Do understand that we are talking about the most general space time variation, and you understand also that this holds somewhat special in the sense when we do such integral fourier Laplace integral; that is somewhat different. Then what you have done in Maths course, there you always take alpha is real, omega is real, why because you have not bothering about growth or decay of these quantities that is what it means. We have seen yesterday also when we were looking at time variation the fact that omega can have in the imaginary part indicating whether it will grow or decay with time.

So, if I put the imaginary part equal to zero; I am basically studying the stable configuration the very fact that we in this course will be talking about alpha and omega as complex implies that, we are not excluding any possibility, any possibility got to understand. Remember in the initial lectures we talked about what happen to stokes. You obtained some solutions in the physical plain and then when you compare, it experimentally did not say much and then we quoted the landau or lifshitz half quoted remark that, in nature you get those solutions which not only exits and are unique, but also those are the one which are there despite the presence of disturbing influences.

So observation observable quantities are those which also stable. So that is precisely what you do in your main stream courses in math's and physics where you restrain this alpha and omega to be here. Here, we would make them half complex values, and it would be one of our cherish goal to find this combination of complex alphas and omegas. We will see that most of the time it is not feasible, so basically on the left hand side what we have the variables; the dependent variables are functions of x, y and t. So here in the fourier integral we have used x and t, so this amplitude must be function of y that is the only way it can be. You could ask me that why not have a fourier series also in y. The answer to this question is not very straight forward and obvious we have sort of inhomogeneity in the y direction, there is a bounding wall that is why the shear has formed that velocity profile. has come about So our y is not unlimited, it is limited to an extent so is our time also. When we talk about time we must have some origin.

So if I perform time integral, we are talking about that receptivity frame work. If I start an interval or a water tunnel that at the time of starting, I am not only starting the mean flow, I am also starting some kind of a disturbance field and I can study the evolution of that disturbance field. So there is a time origin, time only goes in a positive direction. Look, the moment I switch on the tunnel the way the disturbance evolves does not depend on the past. This may seem a little difficult to understand but you would understand it better if you conceive of situation where you have very clean flow. Disturbances are not affecting it, and now you introduce a source of disturbance in a very deterministic manner. yourself and When you switch on this disturbance source that is your time origin and the events that happen subsequently does not depend on what happen before hand.

So that is what is called causality however, in this case we would not use the fourier transform in the y direction because of inhomogeneity of the flow geometry. We will keep it like this and then we will see what you will see if I introduce two transforms in x and t directions where the flow is two dimensional, and if I have a partial differential equation I would end up with ordinary differential equation in y that is what we would be doing. So let us put this description for the disturbance field in terms of their fourier Laplace amplitude u bar, v bar and b bar. So once we have done that what we need to do is then plot this information into our governing equation.

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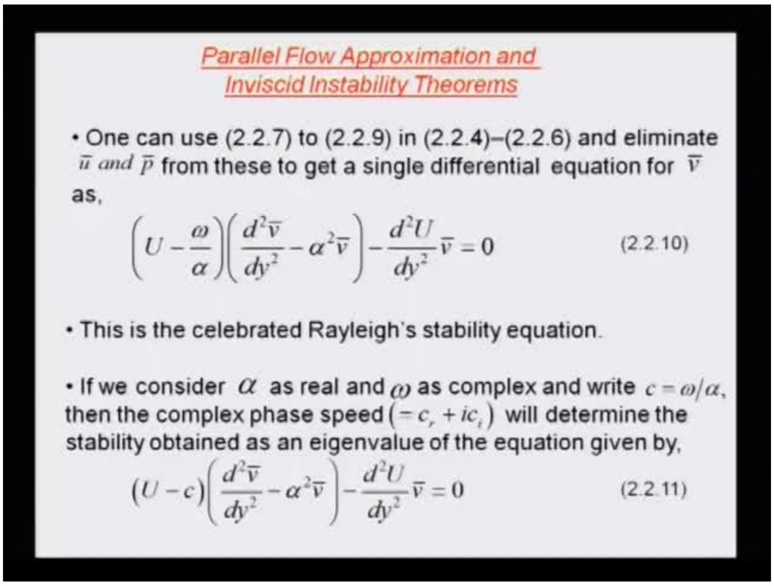

Now this records a little bit of explanation, so let me just do it gently so that you understand once for all how this process is carried out. Let us get back to our discussion on this parallel flow approximation, instability analysis of a two dimensional flow and we did obtained the governing equation. Let me go back in the slide and you can see how it was so we will keep that here.

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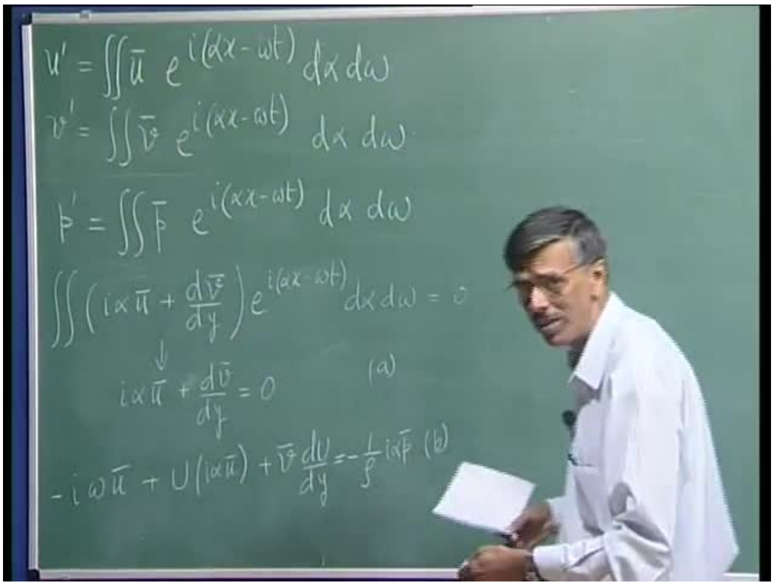

So these are those three equations that we have. Now what we have written down is basically nu prime, add them the double integral u bar e to the power i alpha x minus omega t, and d alpha d omega. So similarly, b prime is v bar e to the power i alpha x minus omega td alpha d omega p bar. p prime should be equal to amplitude p bar times space, all are acquiring at a phase that is a very loose way of saying because if you has an imaginary path it will also contribute to the habituation or growth of the amplitude. The amplitudes are like this that your amplification in space and time would come about from imaginary part of alpha and omega here, so we should keep that in mind. But we have these three definitions in the specter plane that is plug it in one by one.

What do I get if I take del u prime del x, I will get i alpha v bar and this phase relationship. So, I am not going to write down the phase relationship but you can understand it is very much there, so I have del u prime del x should be equal to this. What about del v prime del y, that y variation is embedded in there, so what I would do is I will write that as d v bar d y and then we have all these quantities that we have alpha x minus omega t d alpha d omega and this is equal to zero; the logic follow. Similarly, if that integral is zero for any arbitrary space time variation then to hold to this relation we this quantity must be equal to zero identically for all possible alpha omega.

So this equation then comes out to this. That is why I promise to you that the PDE will be converted into the ODE. Can you not see that what we have is a differential equation where the derivatives are with respect to y alone. What about the two point five if I do it what do I get, from now I am simply worrying about kernel of the integral like this. So del u prime del t will give you what minus I omega e bar. So I will have minus y omega u bar that is the first of del u prime del t. What about the following term, u is the function of y only, so when I write down the fourier transform with respect to x and t it is virtually like a constant.

So, there I will have the capital u sitting there as it is what about del u prime del x del u prime del x again is going to be i alpha u bar, so I will have i alpha u bar so I will have i alpha u bar. What about v prime d u d y that is much (( )) that will be simply v bar d u d y. I am on the right hand side, so what we will have minus one upon rho, and del p prime del x will be i alpha p bar. So that would be there.

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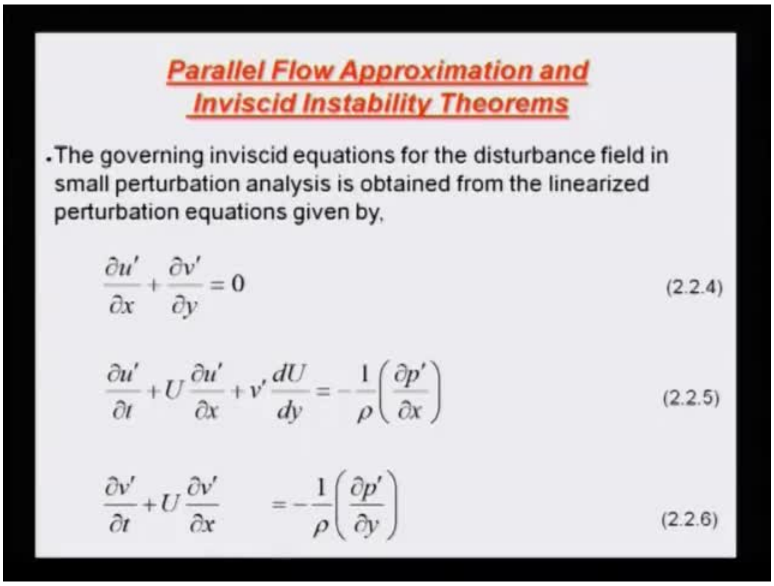

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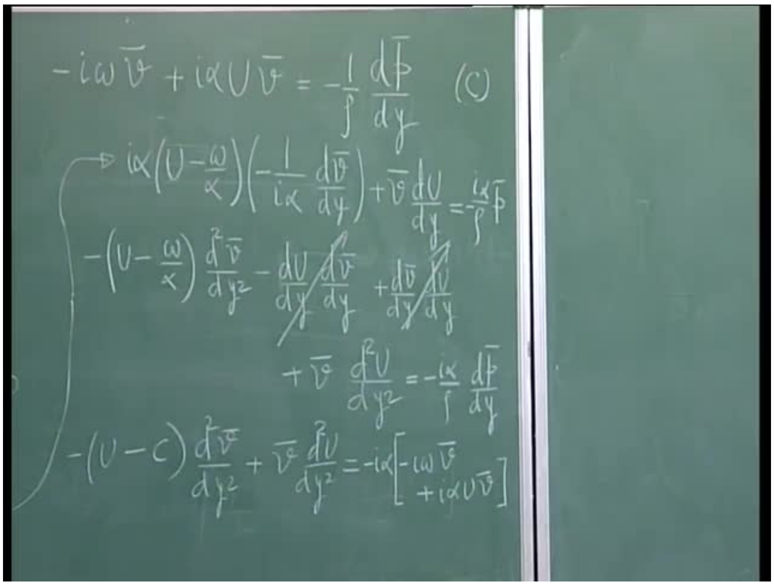

So this is let us say equation a, this is equation b coming from x momentum equation, and what is the last one, it will come from here two point six. del v prime del t will give us simply minus Iomega v bar u plus u times del v prime del x will give us i alpha u v bar. On the right hand side I will have one upon rho, and what about del p prime del y that will be simply d p prime d y. So I have basically three equations for this, three fourier Laplace amplitudes u bar and v bar, and p bar. Now I could use this equation a and I can see that u bar is nothing but minus of one over i alpha d v for d y.

So I could actually put this two together and what I would get, I have two terms at which I can write if I write i alpha then I will get U minus omega y alpha into u bar. See there is a c bar, and from here I have found what u bar is. So I will write that u bar is minus one upon i alpha times d v bar d y; that is the term hum. Then I have the v bar term so that is v bar d u d y and that is equal to minus i alpha by row c bar. So what we have done in a sense eliminated u bar, we have u bar from a and b. Now, between these two equations we can eliminate v bar also, what do I do differentiate this last equation with respect to y and then I have a expression for d p bar d y in terms of v bar, so we can do that. What would we get here, as you can see this i alpha goes of hum, so what do I get minus U minus omega by alpha and we are differentiating with respect to y.

So I will get d two v bar d y square, but note u is also a function of y, so I will also get another term which will be u d u d y into d v bar d y. that we will get what about here

From this I will get d v bar d y times d u d y, and then I have another term v bar d two u d y square that will be equal to minus i alpha by rho d p bar d y. So, you can see these two terms cancel out these two terms and then you get this, you get this expression minus of U. What is omega by alpha, omega by alpha is the phase p; that is the definition of phase p. So I will write it as u minus c, and that I have d two v bar d y square plus v bar d two u d y square is equal to, I will have to get this. So I have one upon rho minus i alpha, and within bracket I will have that expression minus i omega c bar plus i alpha u v bar. So this is simplified form.

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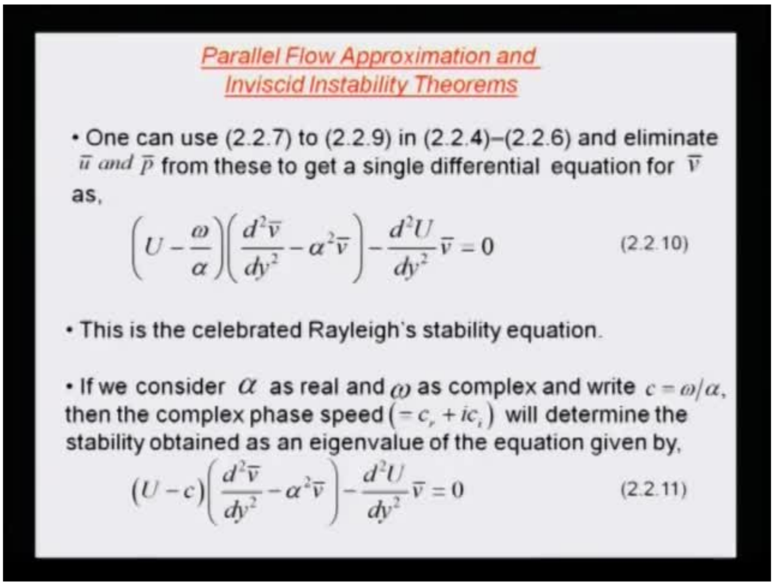

So, we eliminated basically u bar and p bar, and this is what we get that is the equation simplify it simplify it that is what you get. So here we have retained omega by alpha as it is, but that is what you are going to get.

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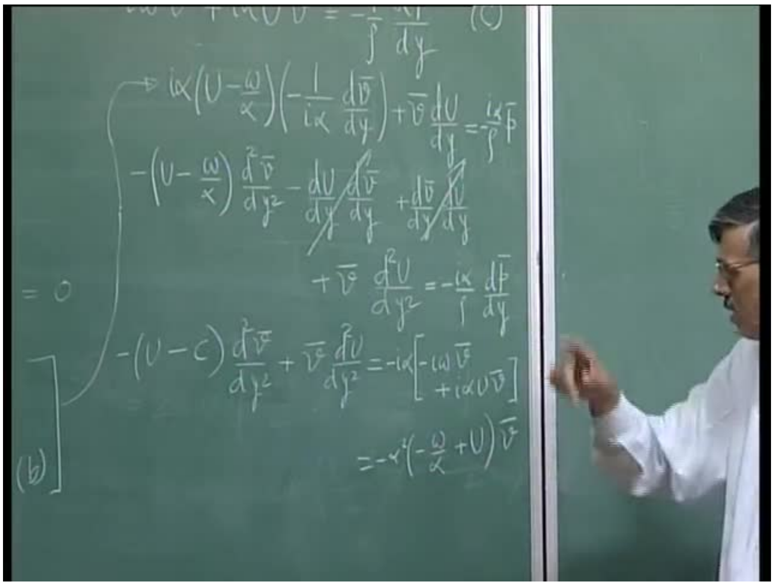

So, you can see here also I can, on the right hand side I could borrowed at i out, and i square will give you minus one. So I will get minus alpha within parenthesis, if I make it alpha square then here I will get minus omega by alpha and here I will get plus u, and that will be multiplied by v bar. So that is what it is, so essentially we are getting u minus e. So that what you have so you can put this inside and then you will find the quantity U minus C multiplied by the second derivative of v bar with respect to y minus alpha square v bar. That is what you are seeing there, so that is the derivation of a equation.

So, once you have that it is very easy for you to look at it. Now, I said some time ago that in stability studies you want to keep the condition as general as possible, and you try to have alpha and omega both complex that turns out to be quite difficult thing. To do so we have to adopt a point of view, what we could do is we could talk about a temporal instability where omega will be complex and alpha will be real, and we will talk about a spatial instability where omega will be real and alpha will be complex.

So, we will start off from this in the next class. We will adopt for this problem a temporal instability approach. I will stop.