Instability and Transition of Fluid Flows

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Module No. # 01

Lecture No. # 04

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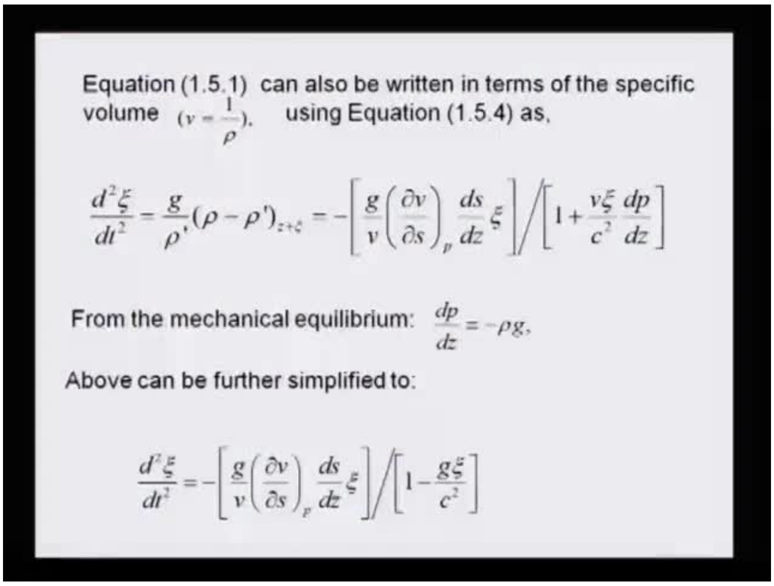

In the last class, we are talking about dynamic stability of a steel atmosphere. And what we arrive at was this differential equation for the displacement xi of a packet, which has been moved all of a sudden from its equilibrium position. And this vertical motion is given by this dynamical equation and that is what we see.

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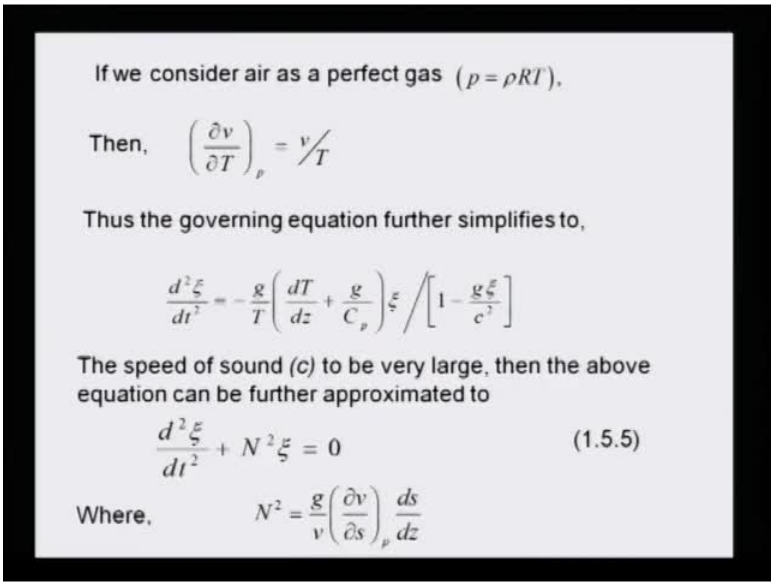

So, we have used thermodynamic relations to simplify the quantity in the numerator, specially the quantity the partial of specific volume with respect to entropy keeping pressure constant, and what we find that such a simplification leads to basically, a simplified equation that is written here in the middle.

So, for a perfect a gas, we have used the constitutive relation to arrive at this. It is customary to really consider this quantity in the numerator or this was the original quantity before simplification as N square. The idea of making it N square would be apparent when you look at that equation 1.55.

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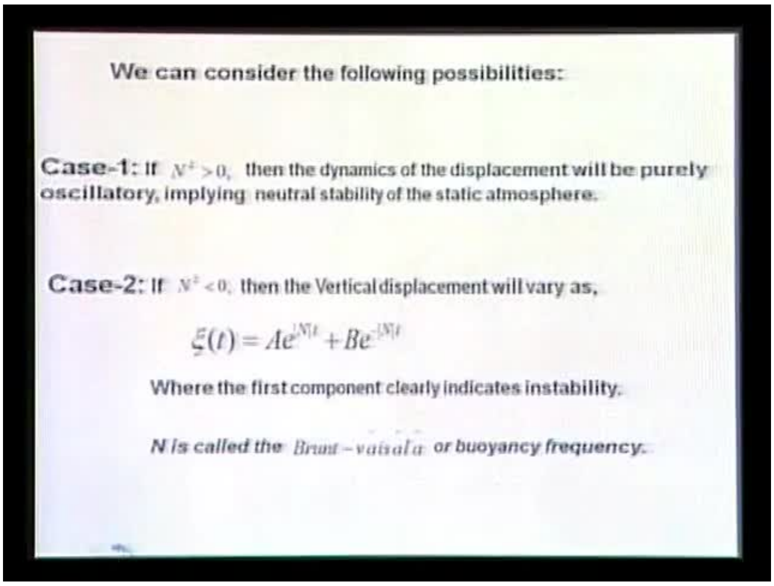

If N square is truly positive, then you can see thus this represents a simple harmonic motion. And that would indicate this vertical oscillation would be neutrally stable because the amplitude will remain invariant with time, although time variation will be given by the frequency N. And that is what was originally done by Brunt and Vaisala and that is what this N is now called the Brunt-Vaisala frequency or the buoyancy frequency.

However, if I take a look at that expression for N, which I have written it down here also in the black board, you can see that it can really take any value depending on what the temperature gradient is dT dz. We also know what is called as international standard atmosphere or ISA, which gives us a statistical distribution temperature variation and which says that dT dz is a negative quantity, following by a rate which is called the lapse rate. And the usual standard that is taken for tropics as well as the temperate climate latitudes that dT dz works out to something like 6.5 Kelvin per kilometer, although we must emphasis that that is a statistical information. If we are trying to look at the stability of a packet, we should look at it value on that particular location, at that particular instant of time. So, there is always this possibility that N square can become negative. And if N square becomes negative, then what you see that the fundamental solutions are e to the power mode N t and another fundamental solution e to the power minus mode N t.

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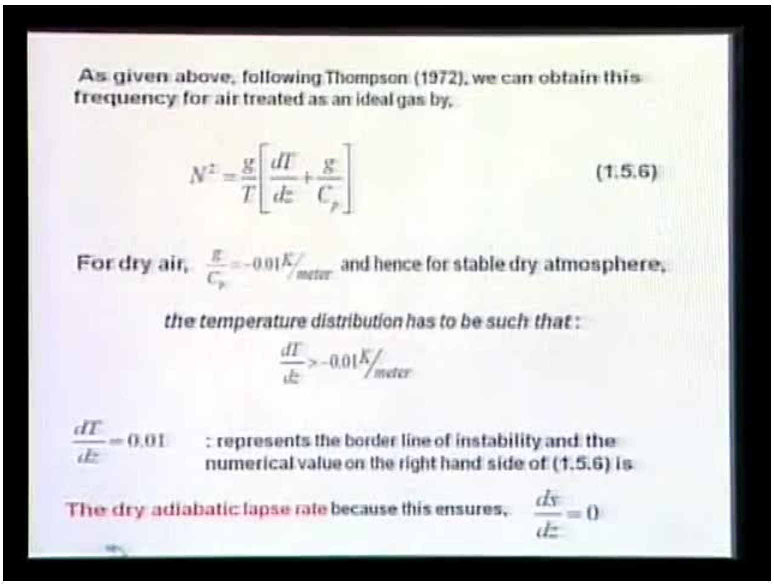

So, you clearly see the presence of this part, the first part would indicate to you that this is going to grow in time. That is the issue. So, what happens is, we need to really look at it a little more carefully that for dry air we know the value of C p and along with the value of g you can find out g by C p works out to something like minus 0.01. And if you really want stability, then this quantity within bracket should be positive, that would require that dT dz should be greater than g by C p.

So, this dT by dz equal to 0.01, this basically represents the border line. If dT dz is less than this than of course, you will have instability. If it is greater than this limiting value, then we will have stability. So, what happens is this value of dT dz equal to 0.01 comes from the expression of ds dz, how the entropy of the ambient air is changing with height. So, this is the reason that this kind of fall in temperature called a lapse rate, this value corresponds to the case where s does not vary with height. So, that is your isotropic case. So, that is why this quantity is called the dry adiabatic lapse rate. Why it is dry? Because the value of C p is calculated for dry air.

So, if you are interested in finding out a particular composition of air, you can estimate the value of C p of a mixture, and then this value may vary somewhat bit, and correspondingly the lapse rate also will vary. So, this is the motivation.

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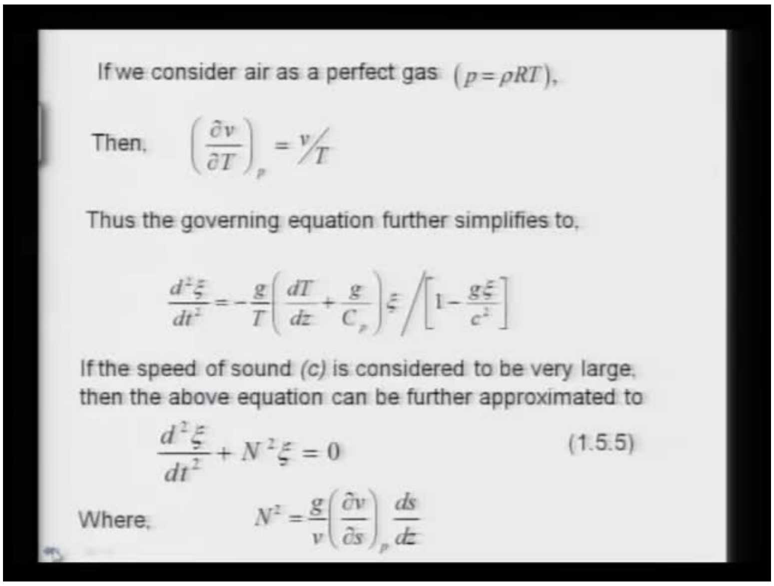

Now, there is one more thing that I would like to point out to you that when we wrote this equation down we could get this kind of an equation, if we consider C to be very large, C is the speed of sound, and if we are considering the ambient air to be incompressible, then the assumption is that C goes to infinity. So, that is perfectly all right. But we know it is not truly so and that is what you have to do is put in the exact value of C a. And then you can see there is some bit of nonlinearity involve because you can transport this quantity in the denominator to the left hand side and you will see immediately the equation has a non-linear flavor.

So, what it basically tells you? The stability or instability would depend on the initial xi that you are also giving because this is a time dependent equation and that probably would not be amenable to your close form analytic solution always, although I think someone has shown me once this. What you notice that in most of your text book, they just simply take C equal to infinity and they just write this equation down. But I just wanted to tell you that this is possible. You should also be aware of the fact that here what is our basic equilibrium state? It is a still air, there is a no convection. What happens if there is a convection? By convection what I mean is there could be some mean motion or there could be even instantaneous fluctuations.

For example, if I create all of a sudden gust on a packet of air, then what happens? I am actually applying a vertical force. So, that would modify by g. So, I could replace that g by g prime. You can understand that how this study can be used or extended to those cases also.

So, do not just simply confine yourself to what most of the text book write as this equation, there are many possibilities. We just now talked about the effect of initial displacement, the non-linearity comes in there, then we talked about if the air is not dry and then we are talking about also what we could do if there is some kind of a vertical updraft anal of a sudden, we can study the stability locally then.

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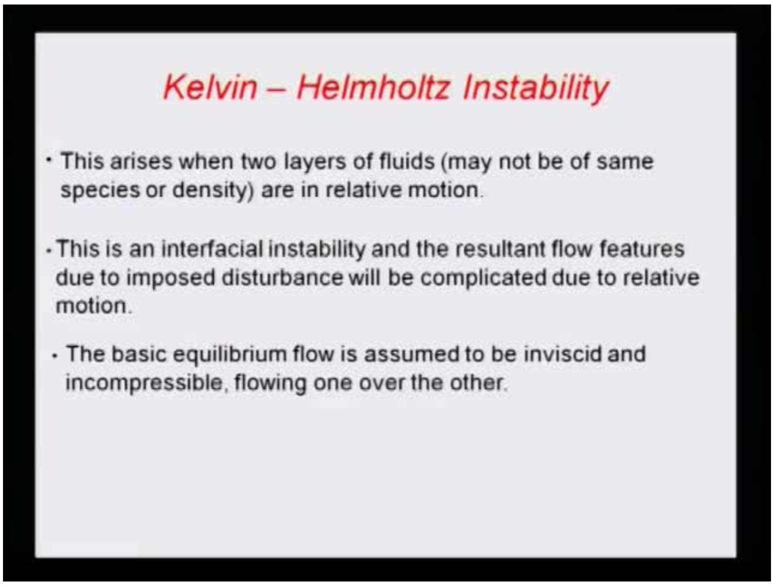

So, what happens is then having arrived here we could perhaps go ahead and study some other situations where we could study stability. So, we have done this and this is what we are going to study.

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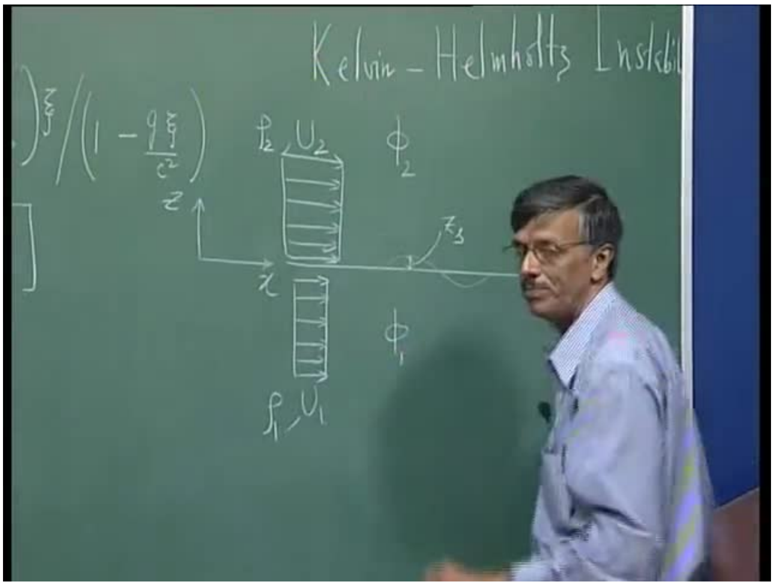

What we are going to study next is basically a new topic and that is what we just now talked about is the Kelvin-Helmholtz instability. What is Kelvin-Helmholtz instability is basically the instability of an interface where there is mean conduction.

So, what does it look like, let me just show you a bit of a sketch, that if I have an interface like this and I can create a coordinate system, let me call the system the z x plane. And here let us say the flow is coming from left to right with a velocity, I call that as a uniform flow as U 2 and on the bottom side, that is say I have a uniform convection that is given by U. You can perhaps visualize it as like what happens on the air water interface on a lake or a sea. So, this is an example. Idealization is the following, that on top we are talking about a uniform flow on bottom we are talking about uniform flow. We want to study the instability of this flow configuration.

So, we are making our task somewhat different. Earlier, what we talked about where U was 0, but here we are talking about some kind of uniform flow. What happens is, whenever you have this kind of a flow configuration, it was Helmholtz who really understood that this interface, if I create some kind of a disturbance at the interface, that disturbance can actually amplify in a catastrophic manner and it could show the instability of that interface. So, that is what we are talking about. Let us say this two layers of fluid that we are talking about also may have a different density.

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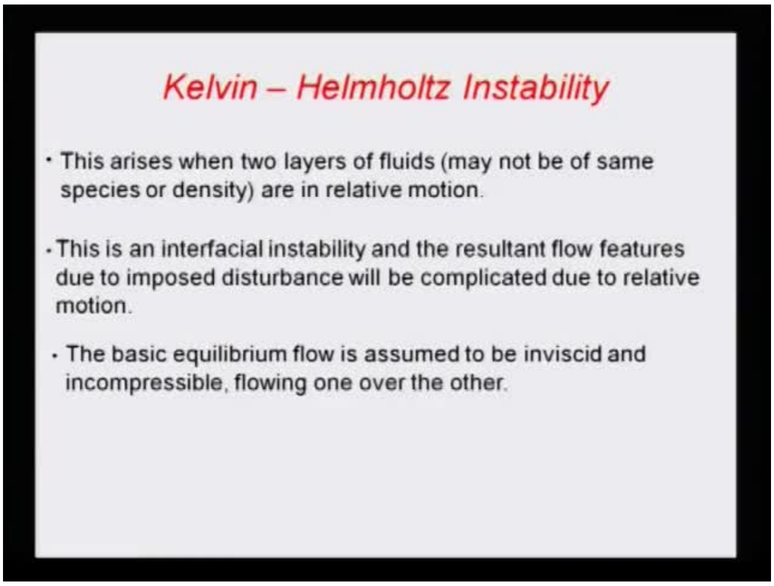

So, we are talking about, say, the case where we have two layers of fluid, they are not of same species, so they have two different density and they are in relative motion. And this relative motion may lead to interfacial instability and the resultant flow features due to impose disturbances are going to very complicated as we will see.

Now, you all is at the stand that when we defined instability, we always talked about instability of what? Instability of equilibrium flow. In the previous case that equilibrium flow was still atmosphere, so there was nothing. In this case, the basic equilibrium flow is inviscid, that is what we are assuming this flow to be uniform on either side. We are also keeping our attention confined to incompressible flow and this two flows are sliding over each other and we want to see what happens next.

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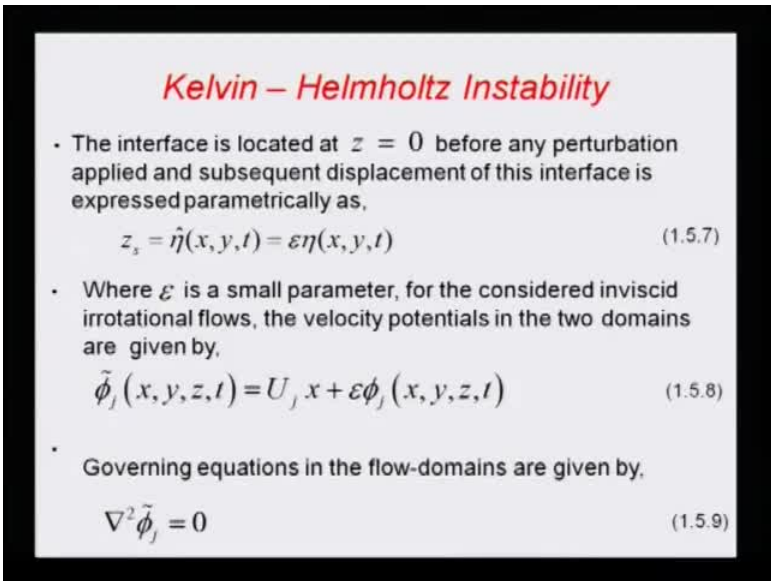

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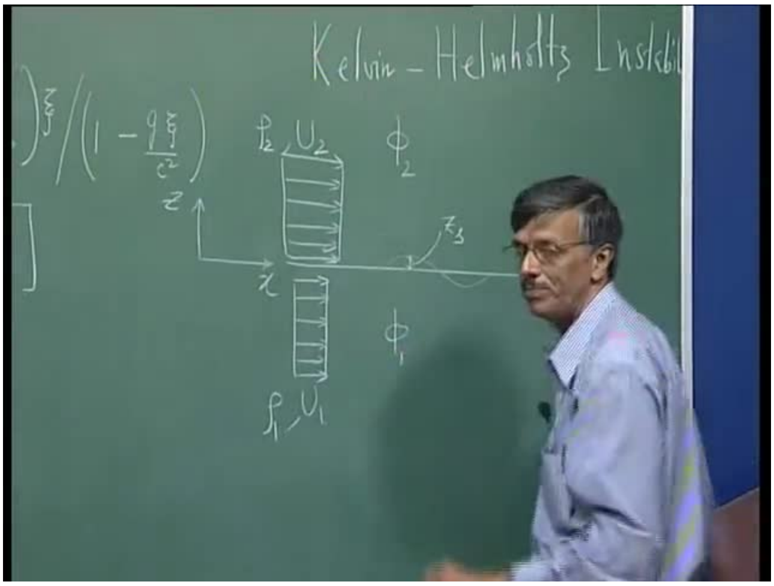

So, this interface that we have between these two fluid here, at time t equal to 0 is flat. Then what we are doing, we are going to give it some kind of a disturbances. And that disturbance quantity, this height over the mean path, this quantity is what we are calling as Z s, that is what your equation 5.7 indicates. And that is going to be a kind of a. Now, think of the flow as a following, that we have the y plane perpendicular to the plane of the bone. So, what is going to happen? We are talking about the interface in the x y plane. So, at t equal to 0, this is definition of the interfaces z equal to 0. But later on, subsequently, it will be a function of x y and t. And let us also keep our formulation simple by considering a small amplitude disturbance. So, that smallness of the disturbance amplitude is prescribed by this quantity of epsilon. We will keep that as a small parameter. And then what is going to happen, that since we are considering inviscid flow, if we also consider it to be irrotationa,l then we can prescribe a velocity potential phi. And we understand that the velocity potential on top and velocity potential in the bottom will be different. So, on the top let us call that as phi 2 and on bottom will have phi 1. Since we have a uniform flow already, so I know corresponding to that uniform flow del phi del x is equal to U 1 or U 2.

So, that is why I could write out a single equation by writing it as phi j with a tilde, that should be equal to U j times x. Now, what has happened? It is a linearized approach, so I could superpose solution. So, I have the basic equilibrium flow given by this is the first part and to that I will say that there is a proportionate disturbance potential occurring in each phase, which I will call is phi 1 and phi 2. And since that we are talking about a small parameter displacement, so I multiply this also by, scale it by epsilon.

So, what happens is, I can see very clearly that the governing equation for this disturbance potential is simply nothing, but given by the laplacian, because we are talking this will be rotational flow, that is what it is. You take it is a second derivatives, this does not exists. So, this part is automatically a solution of phi. So, phi tilde del square phi tilde is equal to del square of this quantity, that is what we are going to see. So, let us now go ahead and see what we can do.

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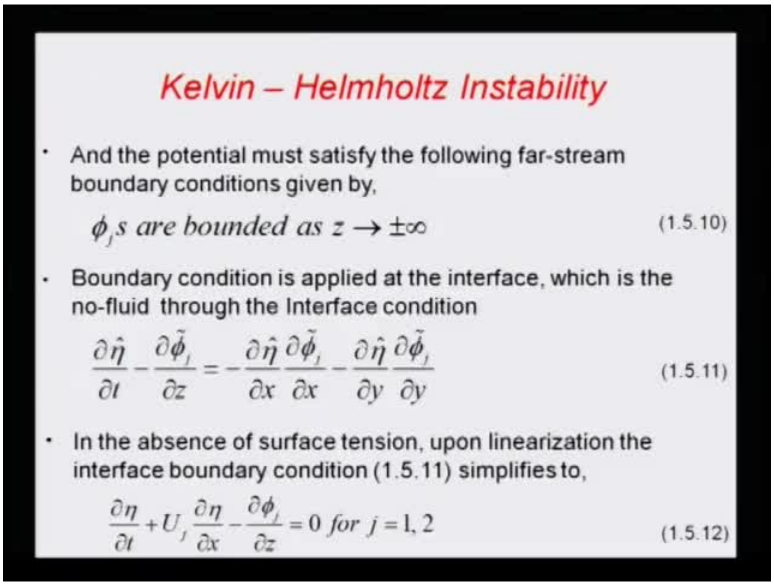

Now, if I want to solve the governing differential equation that we have just now seen , need the boundary condition. What are the boundary segments here? One segment of the boundary is the interface and another segment of the boundary is infinitely above and infinitely below, because we are talking about an unbounded flow separated by that interface.

So, that is what we are talking about, that z can go to plus infinity on top and minus infinity at bottom. And because we are talking about a physical system, the disturbance energy is going to be bounded. Where does the disturbance energy come from? It comes from the equilibrium flow. And if the equilibrium flow has a finite energy, disturbance flow also should have a finite energy. So, that is the whole idea, that disturbance potential should remain bounded. Please do understand boundedness and putting equal to 0 are not necessarily the same. So, this is going to help us. Now, let me spend a little time and tell you little more about the other boundary condition, which is the one at the interface.

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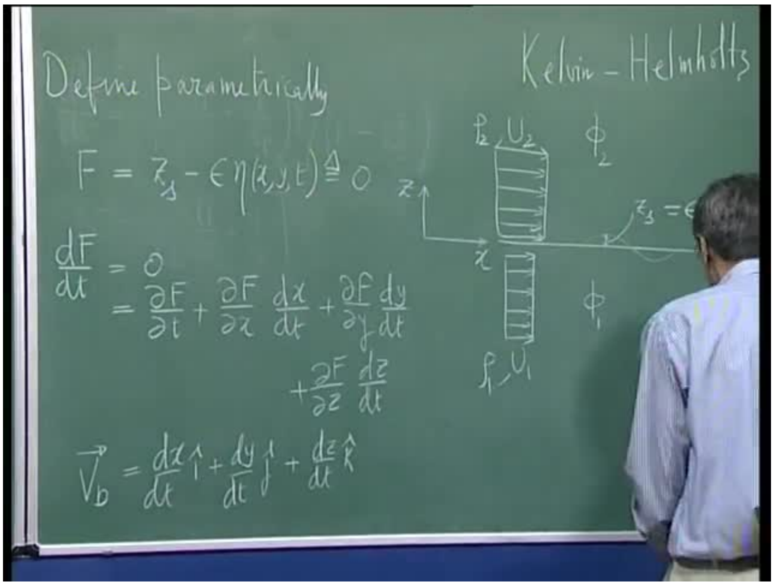

So, to do that, what we are going to do? We are going to use our concepts of kinematics of flow and let us see where we can. See, suppose this interface that we have given as what, epsilon eta as a function of x y and t. So, I could parametrically define the interface by this F, which I will write it as Z s minus epsilon eta x y. So, this is equal to 0 that is the definition. So, F equal to 0 is the definition of the surface. If that is indeed the case, then what we could do is, in subsequent and what will happen to this interface, because these are not two miscible fluid, they are not going to mix with each other. So, subsequently at all time its total derivative should be equal to 0. And what do you get this total derivative as? Del F del t plus, well you know what we will have to do, we will have to take the convective path. So, that I will write it as del F del x. I will write a corresponding path as the dx dt. And similarly, I will write del x del y dy dt plus del F del z dz.

So, basically what we have done? Think of the following that I have taken a dF and that is nothing, but del F del t plus del F del x dx del F del y dy. And then if I take the substantial derivative, then I will get this local part and this. And what are these quantities? These quantities are going to be the component of the velocity of the boundary motion. So, what happens is that boundary motion, I will call it as V b is nothing, but dx dt unit vector plus dy dt unit vector and dz dt the unit vector in the z direction.

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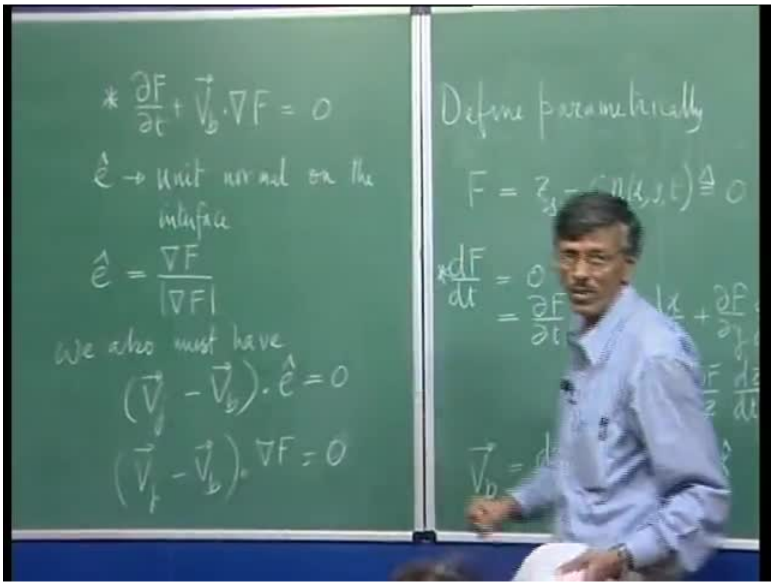

Now, if I look at it, then this equation I could write it as in this form like del F del t plus, then what happens to this, this is going to be V b dot gradient.

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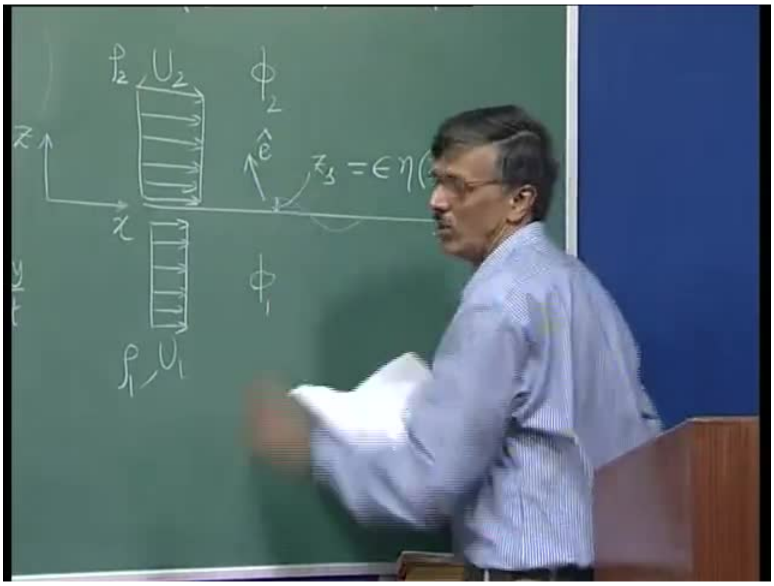

So, this is written mole like a coordinate frame independent quantity. So, we are talking about a coordinate frame independent quantity. Now, if I talk about a unit normal on this surface and I call it like e hat.

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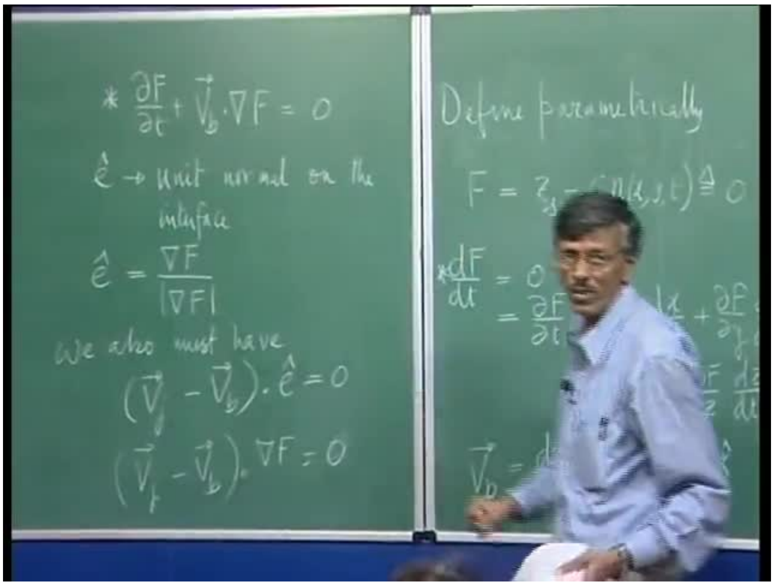

So, e hat is basically the unit normal on the interface. So, what do we get? What is this e hat? We know what this e hat, that is grad F divided by mod of grad F, this we know. This we know from our basic calculus that is what it is going to be. What else do we know that on the interface the boundary motion must match also the fluid motion.

So, what we get actually then, we also have the velocity in the each phase, which I will write it as V j. On the top surface it will be V 2 and the bottom surface it will be, bottom part of that same surface it will be V 1. Then the relative velocity is this, the body motion. And this relative velocity, when I actually take a dot product this must be equal to 0, for what reason? That the normal velocity is 0. e defines the normal, so what we are saying there has to be no 0 normal velocity on either side, otherwise mass conservation will break down.

So, this is what we must have. So, basically then what we are seeing, that this is the relation and then I could use this here because this is a vector. So, what I am getting is V j minus V b dot product and this equal to 0, because the modulus of that I can put it on the right hand side and this is what it is. So, what we are seeing her,e that V b dot del F should be equal to V j dot del F.

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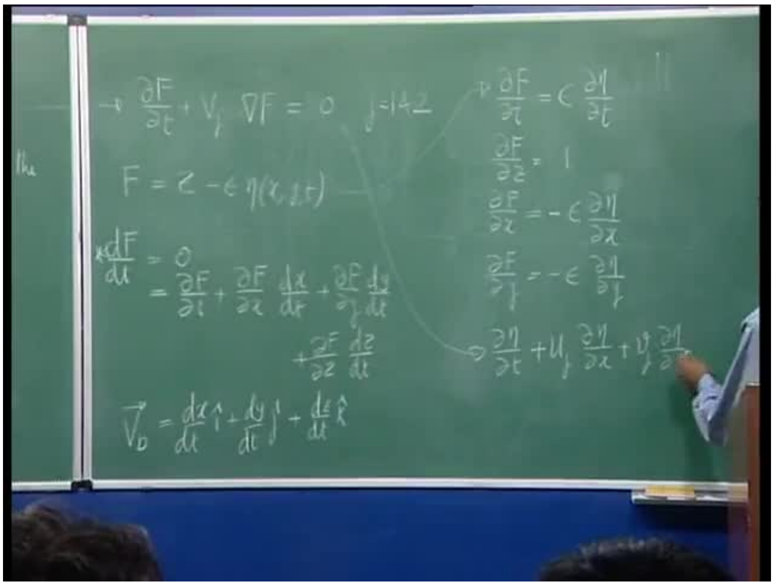

So, this is what we are going to use. So, this equation that we have written here is going to be something like del F del t. So, this part I could write it as del F del t plus V j dot grad F equal to 0. And this j actually corresponds to 1 and 2.

So, we are basically from one boundary condition we came out with two boundary conditions, because we have to solve the problem in this two different side of the interface separately, so we need adequate number of boundary conditions. So, this is what we are going to look at and I will just simply erase this. And what we are going to see then, basically we have two sets of boundary condition.

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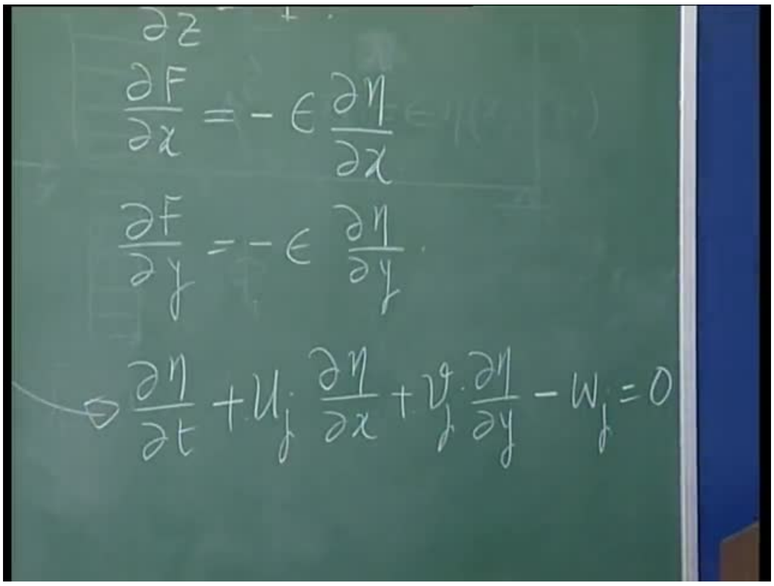

No what is our F? We have just had it here, F was nothing, but Z minus epsilon eta x y t. So, from there what we find, del F del t would be what, could be equal to epsilon del eta del t. And what about del F del Z? 1.And del F del x is going to be minus epsilon del eta del x. And del F del y equal to minus epsilon del eta del y.

So, basically then this condition would be what? The first quantity is del eta del t. I am going to cancel epsilon everywhere, I will do that. Then, what I am going to get? What about this part? Then I will get the u component of velocity times del F del x. What is the u component of velocity? I will write that as, for the time being, I will write it as u of j. That I will get it as del eta del F. Well I have epsilon everywhere, so that is what I am not writing this and v j will be del eta del y. And I will have W j and since I have taken a minus sign everywhere, so what I am going to get is, get this as equal to minus W j is equal to 0. So, this is going to be my boundary condition for both the phases, I will note down the values of the different j's and will work it out that way.

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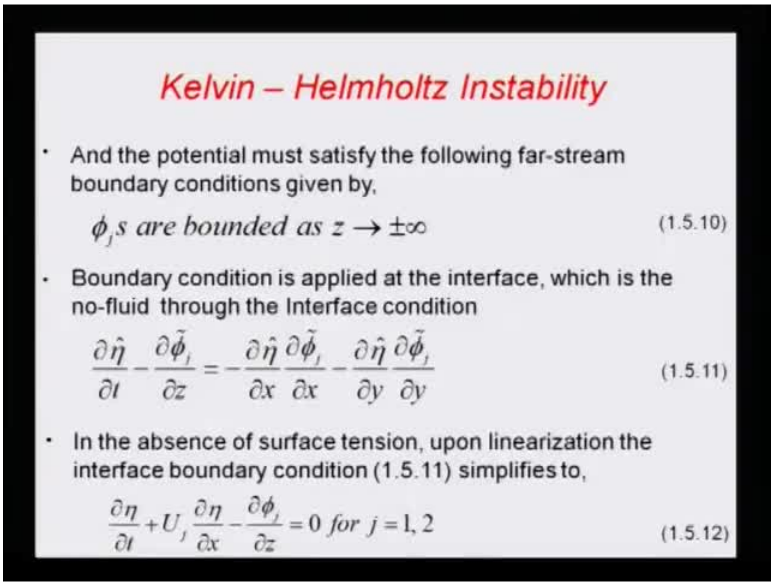

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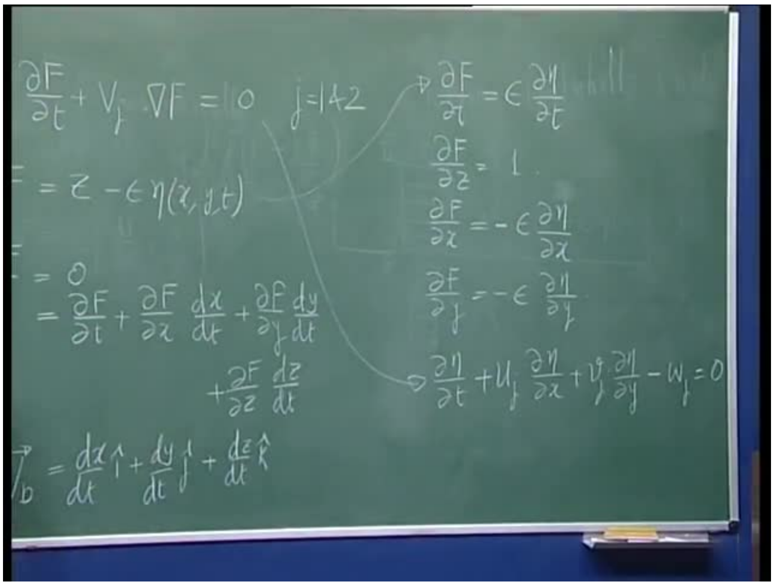

So, that is precisely what you see in the black board here, then 1.511 is how it is derived there. So, this is the detailed derivation. Now, what you notice that these quantities, now I have taken out epsilon, but what about u j? u j has two components one coming from capital U j the mean motion plus the disturbance quantity. So, if I am being consistent, if I am writing the quantity of the same order, then I will not write that del phi j del x times this quantity that will be a lower order quantity.

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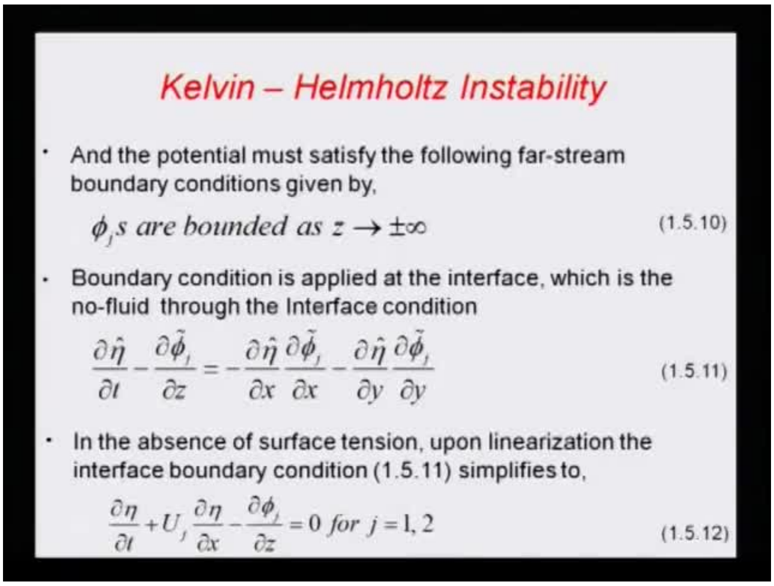

So, what happens is this actually done can be simplified in what we have written it there.. So, what you find then that this quantity is what? This is a small quantity multiplying by another small quantity. So, if I have skipped a few step, let me fill it up and explain to you cleanly.

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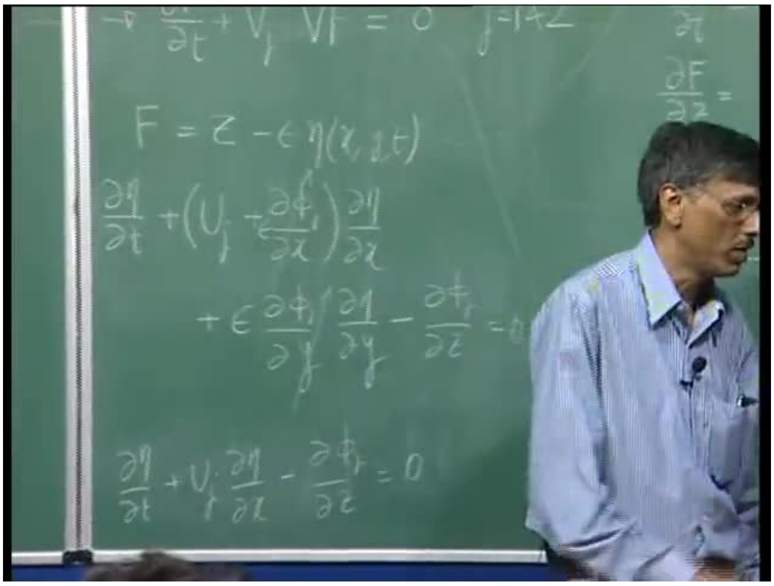

Now, what we are talking about here is, if I write this equation down del eta del t this and then we are talking about U j plus, basically we are going to write del phi j del x, that is your U j, this whole thing is U j. And this has a epsilon belted him. And then we are multiplying it by del eta del x. What about the other quantity? I have epsilon del phi j del y, that is your v j times I will write del eta del y. And what about W j? W j is nothing, but del phi j by del z.

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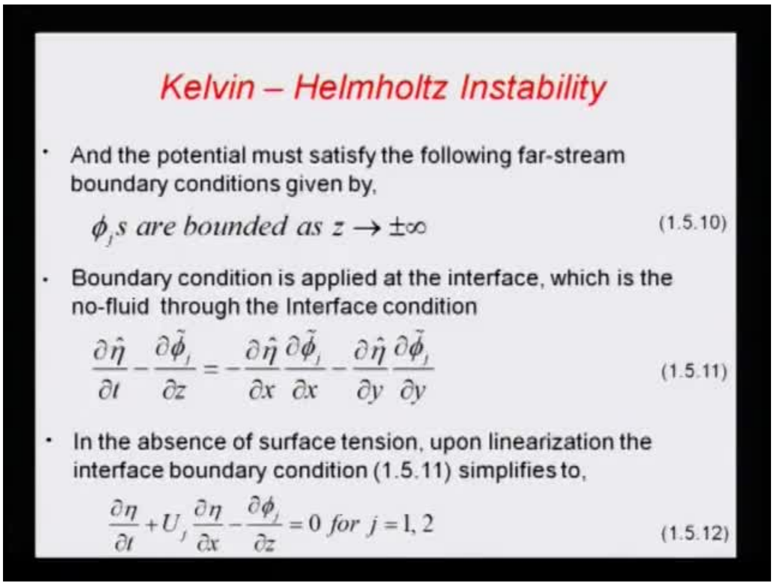

Now, you can see that if I just write down the order 1 quantity, then this term will go away, this is order epsilon quantity, this is lower order quantity. So, what I get is del eta del t plus U j del eta del x minus del phi j del z, and that is what you see there over there on the black board, that is what we have written down there. So, this is applying the boundary condition and linearizing. We have removed epsilon square term, we have kept the term only up to order epsilon, that is what your 5.12 is. And then what we could do is, we could do something more.

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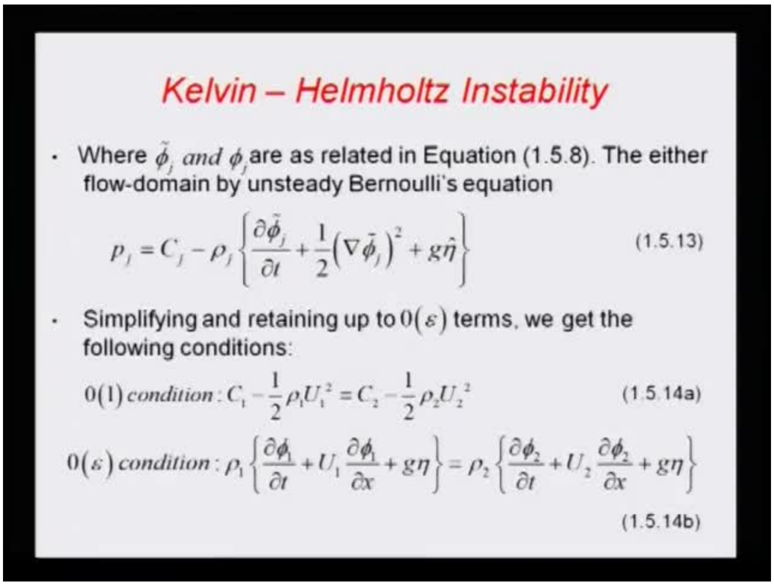

Now, we have gotten the boundary condition here at the interface like this. What else we need to ensure? We need to also ensure the pressure at the interface, I have some pressure on top, some pressure at the bottom, how do I basically work out that quantity? So, that also requires a little bit of a knowledge of our knowledge equation written for is unsteady motion. See, basically now although we are talking about inviscid incompressible flow and we are taking about, for such a flow the Bernoulli's equation also needs to be somewhat modified because of the unsteadiness.

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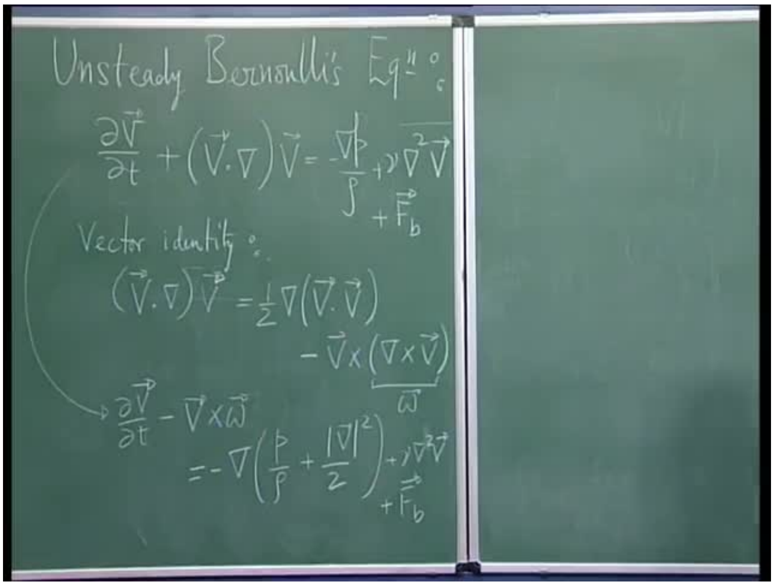

How do we do it? Well, let me just spend a little time explaining to you how unsteady Bernoulli's equation could be obtained, that is what we have written there, but let us explain it somewhat better, I will give you a quick derivation of it starting from the Navier stokes equation.

A Navier stokes equation gives us the following, V dot grad V and on the right hand side you have gradient of pressure (( )) rho, and then you have nu times, there is viscous diffusion term and let us say we also have some kind of a body force. So, this is what we have. Now, let us use a vector identity. So, this is an essentially a vector identity, it holds for any vector and we going to use it for the velocity, which tells us that V dot del times V is nothing, but equal to half gradient of V dot V minus V cross del cross V.

So, this is something like your gradient of V square by 2, something like specific kinetic energy. So, we can substitute it here. If I do that, what I could get is the following on the left hand side, I will keep the convective local oscillation term and I will keep this term. What about this del cross V is the vorticity, omega. So, we can keep that on this side. So, that is going to be V cross omega. So, this is your omega vector, vorticity vector that is what we are going to get. And let me transport this term on the right hand side, I already have a form of this kind, gradient of p by rho. And this term, when I have taken it on the right hand side, that also will have a minus sign. So, I will get there as this, plus what else is there, we have this term nu del square V plus a body force. Now, let us make some simplification based on what we are doing currently here. What we are doing currently here is basically looking at inviscid irrotational flow.

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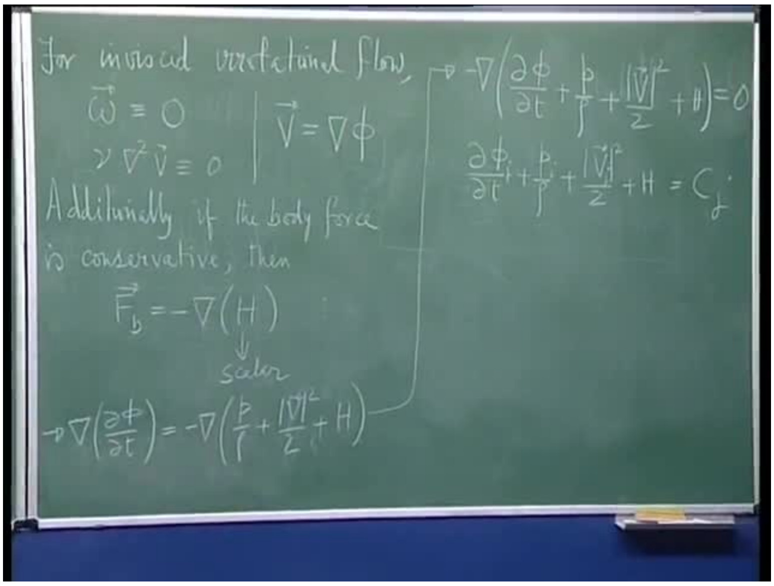

So, what happens? For inviscid irrotational flow, what you can do? Irrotational flow, so what is this, identically 0. And the inviscid means the viscous diffusion term is also equal to 0. So, I will write that term equal to, identically equal to 0. We have written a body force now. Suppose, I write it like this if the body force is conservative, then how can I write a body force as? A conservative body force would be minus a gradient of a scalar quantity, let me call that as some scalar quantity H. Then what do I get from here, you can see that we have this and because it is irrotational flow, so potential exist, V itself is going to be take grad into 5. So, the first term on the left hand side is going to be del del t of gradient of phi, I can interchange the time derivative operation with the spatial gradient operation and that would give me and I put it on this side.

So, what I am going to get. So, let me write it down. So, I am going to get gradient of del phi del t, that is that term. So, this term has gone and on this side I have gradient of p by rho plus… And a V is a minus, so I can put that equal to H. So, what we could do is, now we can transpose this term also on this side and then I am going to get the gradient operator operating on del phi del t plus p by rho plus V square by 2 plus H that is equal to 0.

So, what does it tell us? If the gradient of that quantity is equal to 0, that means that quantity must be equal to constant anywhere I look at in the flow field. So, that is going to be your unsteady Bernoulli’s equation that we started looking at it. Now, because we have two phases, one on top, one at bottom. So, what I would get, I will write this as this and this as this and this will be corresponding V j.

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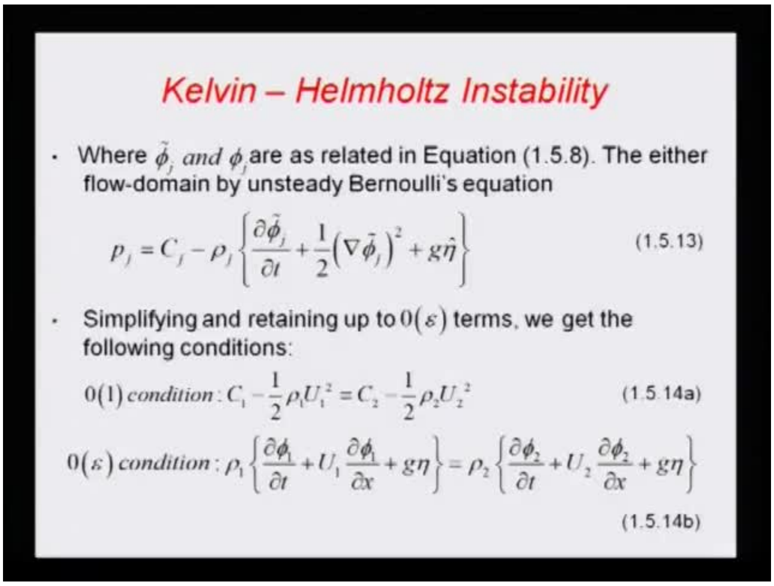

So, if we basically talk about p j as the pressure on either side of the interface, then we get that, that is what we have written it down here. Because now you can see here the body force is due to gravity, so the interface is at deflected by height say, epsilon eta. So, the r eta cap. So, then I get capital H is equal to g eta. So, that is precisely what we are going to use on either side.

Now, of course, we have seen what we have defined, the pressure must be continuous. If I neglect surface tension, if the surface tension is not considered at all, the pressure must be same coming from top as well as going up from the bottom. So, if I equate this, then I will get a set of terms. One would correspond to order 1 term, order 1 term will have this quantity. And what about here? It will have an order 1 term, on the lower surface it will be half rho 1 U 1 square. And on the upper phase, I will have the constant C 2 and minus half root U 2 square. Please do understand there is constant has to be different, because we having two different phases, that is order 1 term and order epsilon term would come from rest, that will be lower side I will have rho 1 del phi 1 del t plus order epsilon term what do I get, that is capital U j plus del phi j del x. So, if I only keep the order epsilon term, that will be the two terms that quantity and there is a half. So, that is what we get.

So, this order epsilon term and from here I get g eta. And the same way I will write it there. So, you see that how this thing can be really taken care of.

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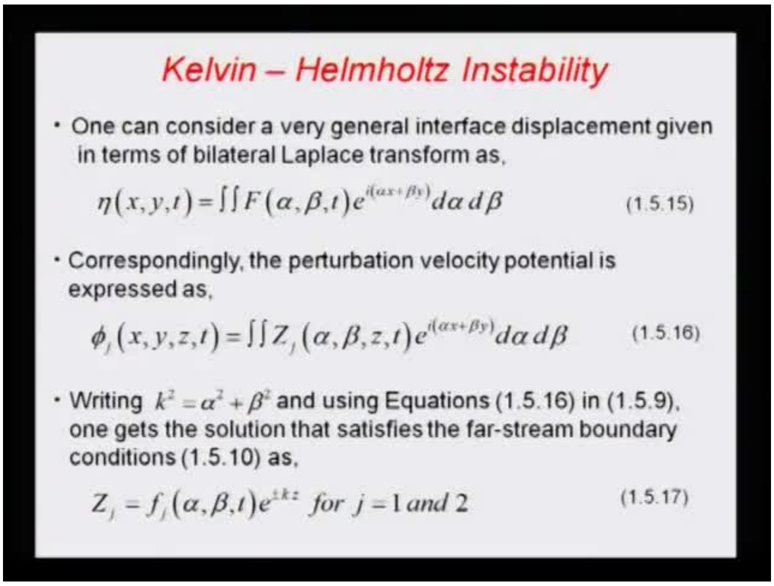

So, now we are not done yet, because so far what we have done here, we have just talk about a general displacement eta. Now, if I talk about general displacement eta, I could also write it in terms of its harmonic component and that is precisely what we have done here, what we are doing it is the x y plane. So, if I write it in terms of its Fourier Laplace amplitude capital F times of a function of the wave number in the x direction, the wave number in the y direction and the time is of course, there and then this is how we should write it over, integrate over, the whole possible ranges of alpha and beta, and that is what this integrals stands for.

So, we are basically talking about very general prescription. So, this does not require any kind of approximation here. The same way what we could do is, we could also basically talk about the velocity potential also in terms of its Fourier Laplace transform, which I am writing it here, amplitude is capital Z which will be now function of x y z and t.

So, what we are talking about. We are talking about its harmonic content in the x and y direction by its alpha and beta, z remains. Remember, why did z did not come here? Because z is small, z was 0 at t equal to 0, that is why we are not talk about it, we are talking about z itself equal to this. So, this whole righten inside itself is z. So, that is what we will have to understand that, this is what we do.

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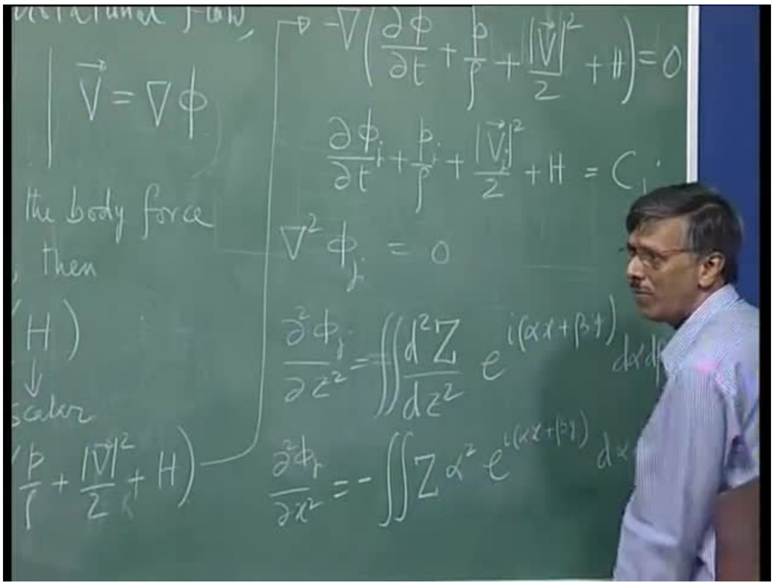

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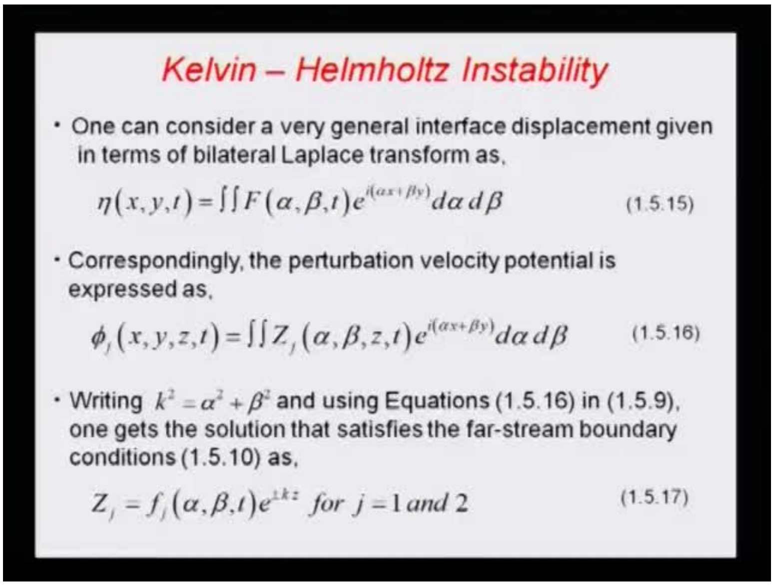

Now, of course, I have the governing equation is as what? Laplacian. If I have the laplacian written down, then what do we get. So, I will be looking at, let us say this term. If I look at this term, what do I get? You can see from 16, z dependence only comes there, so that for a given alpha and beta this I could just simply write it as something like d 2 capital Z d z square. And of course, what you are going to write, those two other factors. One corresponds to the phase e to the power this times the area in the spectral plane, that is d alpha d beta. So, that is what you are going to get. What about a term like this? I will get, if I differentiate that with respect to x in this quantity, I will get i alpha here, for the first derivative. If I do the second derivative, it will be i alpha whole square. So, that I will get as minus alpha square. So, that is what we are going to get. We are going to get a minus sign and then I will have this capital z times alpha square, minus sign has been put outside, so there is no confusion. And you can say similarly I could also write down del square phi j del y square, that will give me minus beta square.

So, when I am going to write it all up there, this x and y derivatives will add up to z into alpha square plus beta square. So, I just could basically talk about the sum of that squares as k square. Physically what is k? See, we are talking about interfacial perturbation which has a wave number in the x direction alpha and in the y direction beta. So, what is the net resultant?

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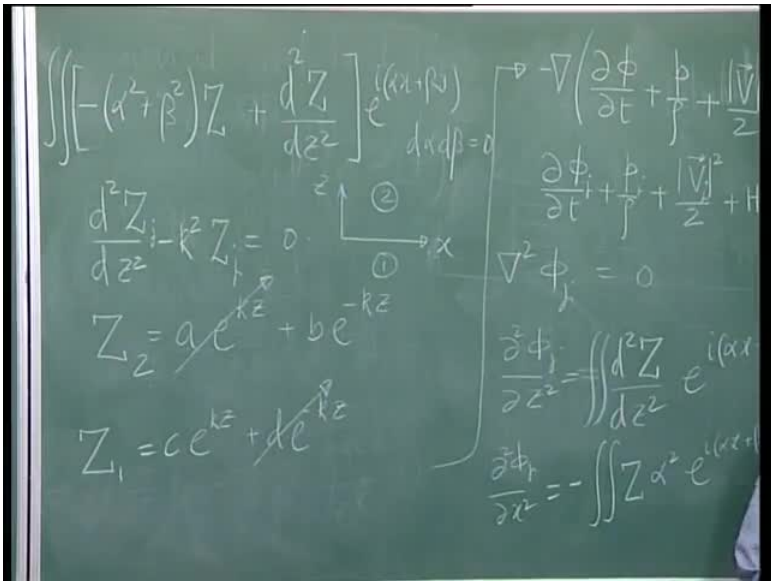

Resultant would be in the k direction, that is what we are talking about. So, if I have this is alpha and this is beta, so i alpha plus j beta would be in the k direction. So, that is what we are seeing. So, if I now substitute all of this in this equation, then what do I get? I will get the following equation, that I will have integral and I have minus alpha square plus beta square into Z and thus second derivative with respect to Z will give me this quantity d 2 square Z dz square this times e to the power i alpha x plus beta y d alpha d beta 0.

So, if this integral is equal to 0, the integral must be equal to 0. So, that is what we find is a governing equation for this amplitude as a function of Z. And you can see for each phase I will just simply add this. So, on top surface I will put Z equal to 2, on the bottom surface I will put j equal to 1.

So, this is a very simple equation, you can write it in terms of a e to the power, let me write Z 2 first on top surface, that will be a e to the power, there will be plus minus, so plus k times z plus b e to the power minus kz.

So, basically what we are talking about? We are talking about x axis here and on this side I have phase 2, on this side I have phase 1. And what is my boundary condition? And now working on this part of the domain, that when I go far away Z go into infinity, solution is bounded. So, what do I expect from this solution? Z going into infinity, so a must be equal to identically equal to 0. So, this part goes away. So, that is your Z 2.

Similarly, I will get for Z 1. Similar two factors which I can write in terms of c and d. So, c e to the power kz and plus d e to the power minus kz. And this solution is valid for negative z. So, that is what happens, when Z goes to minus infinity, this term must go away, so I get this.

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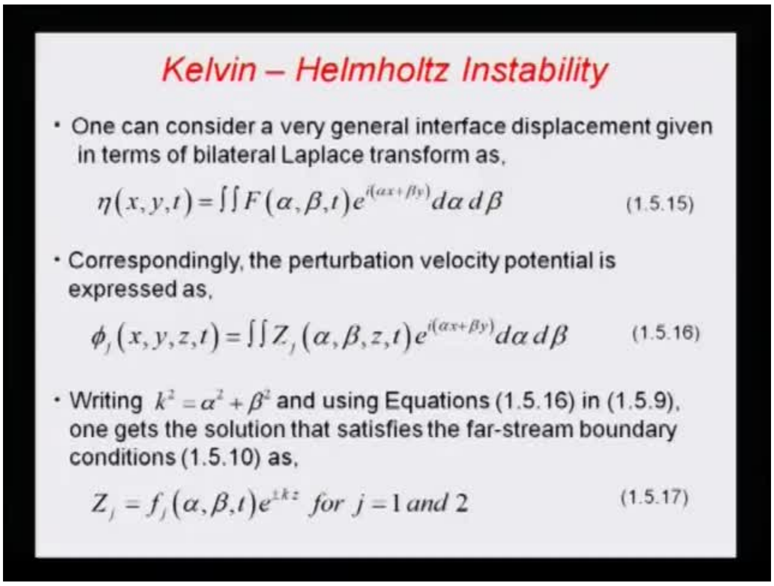

So, what I can do is, basically I could just simply write down this solution together in terms of this which we have done here Z j equal to f j e to the power plus minus kz. So, for Z 1 it will be the plus part I will retain, for j equal to 2 I will keep the negative part. So, that is that. So, you have got it.

Now, I think I will stop here. In the next class, we will further simplify and see how we can use other boundary condition. We still have not used the interface boundary condition, that is what we will have to do.